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STATISTICAL ANALYSIS OF THE FLOW OF HIGHWAY TRAFFIC THROUGH A SIGNALIZED INTERSECTION*

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Abstract. The following is a report on some calculations of the statistical distribution of delay times due to a fixed-time traffic signal on a single lane highway.

In Sec. 1, a model of a traffic light is proposed leading to a set of dynamical equations describing a relation between the times at which cars leave the light in terms of the times at which they arrive. In Sec. 2, some equations are derived for the conditional probabilities that a car will leave at any specified time if it enters at some given time. For this, it is assumed that the time intervals between incoming cars form a set of independent random variables and that one seeks only the equilibrium solutions for which the arrival time of any individual car has a constant probability density.

In Sec. 3, a procedure for obtaining approximate solutions of these equations is derived which actually gives exact solutions for the special case in which the cars arrive at equally spaced time intervals, discussed in Sec. 4. In Secs. 5 and 6 this procedure is also applied to obtain first and second approximations in the special case in which cars arrive with the maximum disorder in spacing possible for this model.

It is found that to a first approximation, it makes very little difference what statistical assumptions are made if one wishes to calculate the average delay.

1. Introduction. As an illustration of how one may apply statistical methods to the study of traffic problems, we consider a relatively simple model intended to simulate the flow of automobiles through a single traffic light on a single lane highway.

The motions of individual cars are described graphically in Fig. 1. The trajectory of each car is represented by plotting its position x as a function of time t. The position $x = 0$ is chosen to be the position of the signal and we represent the state of the traffic light by a dark line for those times during which the light is red and by a thin line when it is green. (Actually the red and green intervals will be effective red and green intervals defined more precisely for the particular model in terms of their influence on the trajectories of the individual automobiles).

We make the following as our initial postulate:

I. The cars approach the traffic light in an ordered sequence, without passing. At sufficiently large distance from the light all cars move with the same velocity v_0.

In view of this assumption, the trajectories of Fig. 1 must approach parallel straight lines asymptotically for $x \to +\infty$ or $x \to -\infty$. No trajectories will cross anywhere.

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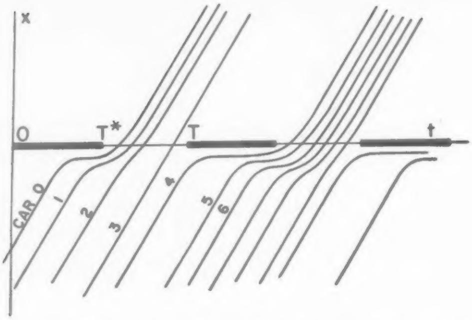

FIG. 1. An illustration of a set of trajectories in which the position x is plotted against t for each car, numbered 0, 1, 2, $\cdots$. Heavy lines represent red intervals of the light at $x = 0$. Light lines at $x = 0$ denote green intervals of the light.

As a result of I, a somewhat more convenient graphical representation of the trajectories is obtained if we choose a "time", $\tau = t - x/v_0$. τ represents graphically a coordinate perpendicular to the asymptotes of Fig. 1. If one plots x vs. τ as in Fig. 2,

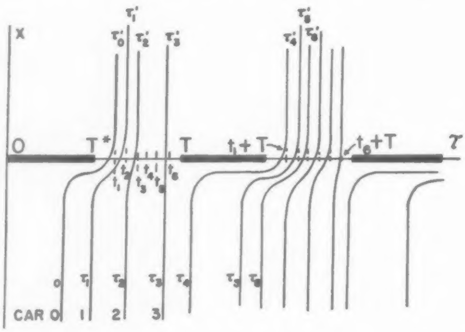

FIG. 2. The trajectories of Fig. 1 are represented by a plot of x vs. $\tau = t - x/v_0$. The "arrival time", the intercept at $x = 0$ of the asymptote of the lower part of each trajectory, is represented by τ_j . The "leaving time", the intercept at $x = 0$ of the asymptote of the upper part of the trajectory, is represented by τ_j' . t_1 , t_2 , $\cdots$; $t_1 + T$, $t_2 + T$ $\cdots$ denote the possible values of τ_j' for delayed cars. Car 3 in Figs. 1 and 2 is not delayed.

the asymptotes becomes vertical lines. At $x = 0$, $\tau = t$. Thus the traffic light itself is represented graphically in Fig. 2 exactly as in Fig. 1. Also the values of τ at which the trajectories cross the light in Fig. 2 correspond exactly to the values of t at $x = 0$ in Fig. 1. For $x \leq 0$, τ represents the time at which the car would arrive at the light if it were to proceed toward the light at a constant velocity v_0 . For $x \geq 0$, τ represents the time at which the car would have left the light had it been moving at the velocity v_0 from the time it was at $x = 0$ until arriving at the position x. Differences in τ along the trajectory thus measure the delay due to the traffic light; the total delay is simply the difference between τ for $x = +\infty$ and for $x = -\infty$.

The detailed shape of the trajectories near the traffic light is of much less practical significance than the difference between the initial and final values of τ, the total delay.

Rather than attempt to propose detailed dynamical laws of motion for each car, we shall propose instead a model which describes only the final values of τ in terms of the initial values and postulates nothing about the intermediate states.

The dynamics of this problem is thus completely described in terms of the variables

$$\tau_j = \text{initial value of } \tau \text{ for the } j\text{th car,}$$

$$\tau'_j = \text{final value of } \tau \text{ for the } j\text{th car.}$$

Since there is to be no passing, we can number the cars according to the values of τ so that

$$\tau_j < \tau_{j+1} \quad \text{and} \quad \tau'_j < \tau'_{j+1} .$$

We shall for convenience describe the values of τ_j and τ'_j as the times of arrival at and departure from the light respectively and we shall further postulate that:

II. $$\tau_{j+1} - \tau_j \geq \delta \qquad \tau'_{j+1} - \tau'_j \geq \delta.$$

Far in front of or behind the light, cars remain at least a certain minimum distance apart. Since all cars move with the same initial and final velocities, we can say equivalently that they arrive or leave at times separated by a certain minimum interval of time δ.

There are certain qualitative features of the motion through the traffic light which should be considered as a basis for future postulates that will yield a detailed description of the relation between τ'_j and τ_j .

Imagine that some car, which we shall number as car 0, approaches the light soon after it has turned red and that all preceding cars have already cleared the intersection. The car comes to a full stop and proceeds after the light turns green. The value of τ'_0 is determined by the time at which the light turned green and is essentially independent of τ_0 . We extend this notion somewhat and postulate the following:

IIIa. If the jth car is delayed in its motion either by the car preceding it or the traffic light or both and if it is the first car to leave the intersection after some red period, then τ'_j is independent of τ_j .

Although the postulate conforms to reality in most respects, it is undoubtedly in some error when applied to a car that approaches the light just before it turns green. Such a car decelerates, but before it comes to a complete stop, the light might turn green and it would proceed through the intersection leaving earlier than if it had arrived somewhat earlier and been forced to stop completely. Such events undoubtedly will in most cases be relatively rare and have little effect upon the overall picture.

In a similar manner we postulate:

IIIb. If the jth car is delayed and is the second car to leave the intersection after a red period, then $\tau'_j = \tau'_{j-1} + \delta$, independent of τ_j . Similarly for the third car to leave, fourth cars etc. We also assume that if all cars leaving during some green period have been delayed, then exactly n cars leave at discreet times τ'_j independent of the manner in which they arrived.

This assumption also contains some error in that the spacing of the cars leaving the light does in practice depend somewhat on whether or not the cars are stopped completely by the light or just slowed down. Also n may depend somewhat on the values of τ_j . To make any more detailed assumptions than these would however make the model extremely complicated and probably add very little to the general picture.

To complete the model we have yet to consider the car that is delayed neither by the light nor the preceding car.

IV. If τ_j occurs during a green period and if $\tau_j > \tau'_{j-1} + \delta$, i.e. if the jth car arrives at the light at least δ later than the j-1th car has left the light, we postulate that $\tau'_j = \tau_j$. A car does not decelerate unless it is forced to do so either by a red light or its proximity to the car in front of it.

Postulates I to IV give essentially a complete description of the dynamics of our model.

In order to express these postulates symbolically, we introduce the following additional notation (see Fig. 2):

T = duration of cycle,

T^* = length of the red interval.

We choose the zero of time at the beginning of some arbitrary red period so that if $\tau \bmod T \leq T^*$, the light is red at time τ.

In accordance with postulate IIIa and IIIb we let t_1 = value of τ' for the first car to leave the light after time T^*, assuming that it was delayed, t_2 = corresponding value of τ' for the second car etc. In general, a delayed car will leave only at the time $t_j + kT$, $j = 1, \cdots, n$, k = some integer, $t_j - t_{j-1} = \delta$.

The dynamical equations of the system expressed by III and IV specify in effect that if car j enters at time τ_j, it will leave at the earliest possible time thereafter. The easiest way of expressing these conditions by equations is to give τ'_j in terms of τ_j and τ'_{j-1}. If we let k'_{j-1} denote the cycle in which the j-1th car leaves the light and k_j the cycle in which the jth car enters so that

$$k'_{j-1}T < \tau'_{j-1} < (k'_{j-1} + 1)T, \qquad k_jT < \tau_j < (k_j + 1)T,$$

then the equations of motion are:

(i) If $\tau_j < \tau'_{j-1} + \delta$, the jth car is delayed by car $j - 1$. This implies that car $j - 1$ has also been delayed for some reason since by II, $\tau_j > \tau_{j-1} + \delta$ and therefore $\tau'_{j-1} > \tau_{j-1}$. We can therefore write τ'_{j-1} in the form

$$\tau'_{j-1} = k'_{j-1}T + t_l \qquad \text{for some } l.$$

τ'_j is then given by

$$\tau'_j = k'_{j-1}T + t_{l+1} \qquad \text{if} \qquad l + 1 \leq n,$$

or

$$\tau'_j = (k'_{j-1} + 1)T + t_1 \qquad \text{if} \qquad l = n.$$

(ii) For $\tau_j > \tau'_{j-1} + \delta$ (jth car not delayed by car $j - 1$) then

$$\tau'_j = \tau_j \qquad \text{if} \qquad \tau_j \bmod T > T^*,$$

$$\tau'_j = k_jT + t_1 \qquad \text{if} \qquad \tau_j \bmod T < T^*.$$

(If the light is green, it is not delayed; if the light is red, it leaves in the first position after the red.)

It follows from the above description, that if one knows the trajectory of one car, it is a rather simple matter, for any particular values of the entering times τ_j, to construct the trajectories for subsequent sequences of cars. By iteration, one simply sends each car in order through free or puts it in the first available "bin" according to the conditions left by the preceding car.

To try to catalog all possible motions for the entire collection of cars is, however, a task of another magnitude. It is a rather uninteresting task also because the questions of primary interest are concerned not with the detailed behavior of the system but rather with the average behavior resulting from some typical distribution of incoming cars. To try to analyze the problem exactly would only yield a catalog of irrelevant information which would probably completely obscure the relevant features of the system. For this reason it is necessary that we approach the problem now from a statistical point of view.

As a final note on the dynamics of this model, it is interesting to note that the equations of motion are not reversible. Although the τ_i' are determined by the τ_i, the converse is not true. Many possible sets of τ_i will lead to the same τ_i'.

2. Some statistical postulates. We shall be concerned here primarily with the problem of determining the distribution of delay times $\tau' - \tau$, assuming that the incoming cars are distributed in such a way that the times between incoming cars $\tau_i - \tau_{i-1}$ form a set of independent random variables. Although much of the study of queues of various sorts have dealt with incoming flows of the above type [1, 2], in fact usually with a Poisson distribution, and have usually considered only questions of delay times or lengths of the queues, these are by no means the only problems of practical interest. The assumption of statistical independence of differences in arrival times is a reasonable one for a long highway free of other obstacles but it is apparent that the flow *leaving* the traffic light is not of this type. In order to study the very interesting problems that would arise when the flow from one obstacle encounters a new obstacle, one must not only relax the restriction on the type of incoming flow that one considers but one must calculate other things than just delay times or queue lengths; in particular, one must calculate the statistical distribution of leaving times. This is obviously a problem of much greater difficulty.

To return to the simpler problem, we concentrate our attention on car 0 as some arbitrary car arriving at time τ_0 in the cycle $k_0T < \tau_0 < (k_0 + 1)T$ and we seek to calculate the probability that this car will leave at a time τ_0' in the cycle $k_0'T < \tau_0' < (k_0' + 1)T$. We let $P_0(\tau_0' \mid \tau_0)$ = probability that car 0 leaves at τ_0' if it enters at time τ_0. τ_0 will be a continuous variable, but except for the special case $\tau_0' = \tau_0$, τ_0' will have only discreet values $k_k'T + t_l$. We also define $P_k(\tau_k' \mid \tau_k)$ = probability that car k leaves at time τ_k' if it enters at time τ_k.

By assuming that $(\tau_k - \tau_{k-1})$ are independent random variables, we guarantee that the problem will be a Markov process of order 1. In so doing, we let $f(y)dy$ = probability that two consecutive cars arrive at times differing by an amount between y and $y + dy$. The function f completely defines the distribution of incoming cars. Actually we have also assumed here that f is independent of k. To do otherwise would be of questionable physical interest and a great mathematical inconvenience. We will not at this time specify any particular form of the function f but it is to be considered as known and to be consistent with assumption II

$$f(y) = 0 \qquad \text{for} \qquad y < \delta.$$

The general procedure of attack on this problem is first to find a series of integral equations expressing $P_k(\tau_k' \mid \tau_k)$ in terms of $P_{k-1}(\tau_{k-1}' \mid \tau_{k-1})$ and $f(\tau_k - \tau_{k-1})$ using the dynamical equations. We then seek to find an equilibrium solution, assuming one exists, by imposing the additional condition that $P_k(\tau_k' \mid \tau_k)$ is not a function of k. In this way we obtain a set of integral equations involving only $P_0(\tau' \mid \tau)$ as the unknown.

We begin the derivation of the equations by considering $P_1(\tau_1' \mid \tau_1)$ for $\tau_1' = k_1'T + t_l$ and $l > 1$, i.e. we seek to find the probability that car 1 will be delayed but will leave at various τ_1' other than the first position after some red light if it enters at τ_1. According to the dynamical equations, this implies that car 0 left at $\tau_0' = k_1'T + t_{l-1}$, $(k_0' = k_1')$. Thus

$$P_1(k_1'T + t_l \mid \tau_1) = \int_{-\infty}^{\tau_1} d\tau_0\, P_0(k_1'T + t_{l-1} \mid \tau_0) f(\tau_1 - \tau_0) \qquad \text{for} \quad \tau_1 < k_1'T + t_l, \quad l > 1. \tag{1a}$$

$$P_1(k_1'T + t_l \mid \tau_1) = 0 \qquad \text{for} \quad \tau_1 > k_1'T + t_l, \quad l > 1. \tag{1b}$$

[We assume here that τ_0 can have any value with equal probability. In general one obtains an equation of this type for the joint distribution of τ_1' and τ_1 in terms of the joint distribution of τ_0' and τ_0. If τ_0 is uniformly distributed, so also is τ_1 since f depends only upon $(\tau_1 - \tau_0)$. Thus one obtains an equation involving only condition probabilities.]

The upper limit of the integral in (1a) is somewhat arbitrary since $f(\tau_1 - \tau_0) = 0$ for $\tau_0 > \tau_1 - \delta$.

The case $l = 1$ and the case $\tau_1' = \tau_1$ must be considered separately. For $l = 1$, car 1 can leave in a first position $k_1'T + t_1$, if either of the following applies.

(i) $\quad \tau_1 < k_1'T \quad$ and $\quad \tau_0' = (k_1' - 1)T + t_n$.

(ii) $\quad k_1'T < \tau_1 < k_1'T + T^* \quad$ and $\quad \tau_0' < k_1'T$.

From this we obtain

$$P_1(k_1'T + t_1 \mid \tau_1) = \int_{-\infty}^{\tau_1} d\tau_0\, P_0[(k_1' - 1)T + t_n \mid \tau_0] f(\tau_1 - \tau_0) \quad \text{for} \quad \tau_1 < k_1'T, \tag{2a}$$

$$P_1(k_1'T + t_1 \mid \tau_1) = \int_{-\infty}^{\tau_1} d\tau_0 \Big\{ \sum_{\tau'_0 < k'_1 T} P_0(\tau_0' \mid \tau_0) \Big\} f(\tau_1 - \tau_0) \qquad \text{for} \quad k_1'T < \tau_1 < k_1'T + T^*, \tag{2b}$$

$$P_1(k_1'T + t_1 \mid \tau_1) = 0 \qquad \text{for} \quad k_1'T + T^* < \tau_1. \tag{2c}$$

The upper limits of these integrals are again somewhat arbitrary.

Finally, for the case $\tau_1' = \tau_1$, we obtain

$$P_1(\tau_1 \mid \tau_1) = \int_{-\infty}^{\tau_1} d\tau_0 \Big\{ \sum_{\tau'_0 < \tau_1 - \delta} P(\tau_0' \mid \tau_0) \Big\} f(\tau_1 - \tau_0) \tag{3a}$$

$$\text{if} \quad \tau_i \bmod T > T^*. \tag{3b}$$

$$P_1(\tau_1 \mid \tau_1) = 0 \qquad \text{if} \quad \tau_i \bmod T < T^*. \tag{3c}$$

Equations 1, 2 and 3 give a complete description of the function $P_1(\tau_1' \mid \tau_1)$ in terms of $P_0(\tau_0' \mid \tau_0)$, assuming that τ_0 is uniformly distributed (consequently also τ_1).

The next step is to seek an equilibrium distribution. This is done simply by dropping the subscripts on the function P so that P_1 is the same function of its argument as P_0 is of its argument. Equations 1, 2 and 3 then become a set of simultaneous linear integral

equations. Rather than rewrite these equations without the subscripts, we shall hereafter use them with this minor alteration implied.

As the equations now stand, we have an infinite set of equations, for determining an infinite set of functions (one for each of the discrete values of τ_k') of a continuous variable τ_1, $-\infty < \tau_1 < \tau_1'$.

This can be reduced to more manageable form by taking advantage of the invariance of the equations to displacements of the time coordinate by multiples of T. Although this invariance does not in itself necessarily guarantee that the only solutions of the equations are themselves invariant to such a transformation, i.e.

$$P(\tau_1' + T \mid \tau_1 + T) = P(\tau_1' \mid \tau_1),$$

such solutions are certainly the only ones that would seem to make sense physically.

In view of this, one need calculate $P(\tau_1' \mid \tau_1)$ only for $0 < \tau_1' < T$, i.e. for $k_1' = 0$, but for arbitrary τ_1. The values of $P(\tau_1' \mid \tau_1)$ for other values of τ_1' are then obtained by using the periodicity.

There is still one other fact that we will use to advantage. Equations 1, 2, 3 are linear homogeneous integral equations. The normalization should eventually be chosen so that

$$\sum_{\tau'_1} P(\tau_1' \mid \tau_1) = 1$$

since these are conditional probabilities. By using this, we can eliminate one of the unknowns [for example, $P(\tau_1 \mid \tau_1)$] thus reducing the number of integral equations by one but by so doing making them inhomogeneous.

3. Analysis of equations. We begin by considering Eq. 1. To simplify notation we let

$$y_1 = \tau_1 - k_1'T,$$

$$y = \tau_1 - \tau_0,$$

and use the periodicity to obtain

$$P(t_l \mid y_1) = \int_0^\infty dy\, P(t_{l-1} \mid y_1 - y)f(y) \qquad \text{for} \qquad y_1 < t_l \tag{4a}$$

$$= 0 \qquad \text{for} \qquad y_1 > t_l\,,\ l > 1. \tag{4b}$$

We can iterate this equation by successive substitutions of the left side into the right side for appropriate t_l until we obtain an expression for $P(t_l \mid y_1)$ in terms of $P(t_1 \mid y)$. By so doing, we obtain the following equation which corresponds to the statement: if car 0 leaves in the lth position, car $(-l + 1)$ must leave in the corresponding first position.

$$P(t_l \mid y_1) = \int_0^\infty dy\, P(t_1 \mid y_1 - y)f^{(l-1)}(y) \qquad y_1 < t_l\,, \tag{5a}$$

$$= 0 \qquad y_1 > t_l\,,\ l > 1 \tag{5b}$$

in which

$$f^{(1)}(y) \equiv f(y),$$

$$f^{(i)}(y) = \int_0^y dy_1 f(y - y_1)f^{(i-1)}(y_1), \tag{6}$$

$f^{(j)}(y)$ represents simply the probability density for a zeroth car and a jth car to be separated by a time interval y. The above integrals are multiple convolutions of the function $f(y)$.

Equation (2a) is quite similar in form to (1a) and by the same reasoning gives

$$P(t_1 \mid y_1) = \int_0^\infty dy\, P(t_1 \mid y_1 + T - y) f^{(n)}(y) \qquad \text{for} \qquad y_1 < 0. \tag{7}$$

Equation (2b) is less pleasant. By using the normalization condition we can avoid considering Eq. (3) and write in (2b)

$$\{ \sum_{\tau'_0 < k'_1 T} P(\tau'_0 \mid \tau_0) \} = 1 - \sum_{\tau'_0 > k'_1 T} P(\tau'_0 \mid \tau_0).$$

The case $\tau'_0 = \tau_0$ is now excluded from the sum because in (2b) we integrate only over those τ_0 for which

$$\tau_0 < \tau_1 \leq k'_1 T + T^*.$$

But $\tau'_0 > k'_1 T$ means $\tau'_0 > k'_1 T + T^*$, since τ'_0 cannot lie in a red interval. Thus $\tau'_0 > \tau_0$ and so the above sum includes only discrete values of τ'_0.

$$\{ \sum_{\tau'_0 < k'_1 T} P(\tau'_0 \mid \tau_0) \} = 1 - \sum_{k'_0 = k'_1}^{\infty} \sum_{l=1}^{n} P(k'_0 T + t_l \mid \tau_0). \tag{8a}$$

Furthermore since $P(\tau'_0 \mid \tau_0) = 0$ for $\tau'_0 < \tau_0$, all terms on the left side vanish if $k'_1 T < \tau_0$; i.e. if a car enters after $k'_1 T$ it cannot leave before $k'_1 T$.

$$\{ \sum_{\tau'_0 < k'_1 T} P(\tau'_0 \mid \tau_0) \} = 0 \qquad \tau_0 > k'_1 T. \tag{8b}$$

In terms of the new notation (2b) and (8) give

$$P(t_1 \mid y_1) = \int_0^\infty dy \left\{ 1 - \sum_{k=0}^{\infty} \sum_{l=1}^{n} P(t_l \mid y_1 - y - kT) \right\} f(y) \qquad \text{for} \qquad 0 < y_1 < T^*. \tag{9}$$

According to (8b), the quantity in the brackets of (9) must vanish for $y_1 > y$. In the final solution, Eq. (8b) and (8a) must be self-consistent and we apparently have the freedom of using (8b) or not here, as we wish. If we do use (8b), we take y_1 as the lower limit of the integral in (9). This leads to a less elegant form of the integral equation but one which is more suitable, at least for approximate calculations. In either case, substitution of (5) and (7) will reduce (9) to a simple form.

Equation (9), without (8b) gives

$$P(t_1 \mid y_1) = 1 - \int_0^\infty dy\, P(t_1 \mid y_1 - y) \left\{ \sum_{l=1}^{\infty} f^{(l)}(y) \right\}, \qquad 0 < y_1 < T^*. \tag{10}$$

Equation (9) with (8b) gives

$$P(t_1 \mid y_1) = \int_{y_1}^\infty dy\, f(y) - \int_{y_1}^\infty dy\, P(t_1 \mid y_1 - y) \sum_{l=1}^{\infty} f^{(l)}(y, y_1), \qquad 0 < y_1 < T^*, \tag{11}$$

in which

$$f^{(1)}(y, y_1) \equiv f(y)$$

$$f^{(l)}(y, y_1) \equiv \int_0^{y - y_1} dy'\, f^{(l-1)}(y') f(y - y'). \tag{12}$$

Equation (10) or (11) expresses the fact that $P(t_1 \mid y_1)$ equals the probability that no previous car left in position t_1. In (10) this is expressed as one minus the probability that a car, l cars ahead left in position t_1, summed over l and integrated over all arrival times of the lth car ahead of the reference car. In (10) this is expressed as the probability that the reference car is the first to arrive after the light turned red, less the probability that if all previous cars arrive before the red light, the lth one left in position t_1; summed over all l and integrated over all arrival times before the red light.

Finally we rewrite Eq. (2c) in the form

$$P(t_1 \mid y_1) = 0 \qquad \text{for} \qquad y_1 > T^*. \tag{13}$$

To summarize, we see that Eq. (5) gives an expression for $P(t_l \mid y_1)$ in terms of $P(t_1 \mid y)$ for all $l > 1$, whereas equations (7), (10) or (11) and (13) together form an equation for $P(t_1 \mid y)$ in terms of itself. The problem is essentially reduced to solving this latter set of equations for $P(t_1 \mid y)$.

The set of equations for $P(t_1 \mid y_1)$ could be incorporated into a single inhomogeneous integral equation with a discontinuous kernel, although this would be of doubtful value. It might be of some help, however, to substitute (11) into (7). Equation (11) gives $P(t_1 \mid y)$ for $y > 0$ in terms of $P(t_1 \mid y)$ for $y < 0$. Substitution of (11) and (13) into (7) would then give an equation for $P(t_1 \mid y)$ for $y < 0$ in terms of itself, namely

$$\begin{aligned} P(t_1 \mid y_1) = {} & \int_0^{T^*} dy\, f^{(n)}(y_1 + T - y) \int_y^\infty dy'\, f(y') \\ & + \int_0^\infty dy\, P(t_1 \mid -y) \Big\{ f^{(n)}(y + y_1 + T) \\ & - \int_0^{T^*} dy'\, f^{(n)}(y_1 + T - y') \sum_{l=1}^\infty f^{(l)}(y + y', y') \Big\} \qquad \text{for} \qquad y_1 < 0. \end{aligned} \tag{14}$$

The above set of integral equations is rather disagreeable in any form. At first one might be strongly inclined to express everything in terms of Laplace or Fourier transforms. This would certainly lead to very simple expressions for the convolutions in Eq. (6) and (12) and even for the sums over l in (10), (11) and (14). One also observes that Eq. (7) would be invariant to displacements in y_1 except for the fact that (7) is valid only for $y_1 < 0$, a condition that is not invariant to such displacements. Similarly Eq. (10) and (13) "almost" have this translational symmetry. These supplementary conditions are just enough to make this approach rather unpleasant and as yet no way has been found to overcome certain difficulties. The other forms of the equations, namely (11) and (14) show this lack of symmetry more clearly.

It seems quite unlikely that one will be able to obtain even a formal closed form solution of these equations except possibly for special functions $f(y)$. One can, however, obtain a useful approximation procedure based upon the assumption that the average number of incoming cars per cycle is small compared with the critical number n.

In the limit zero flux, no interaction between cars, $P(t_1 \mid y_1)$ must obviously be either one or zero accordingly as y is in the interval $0 < y < T^*$ or not. In the above formulas, this approximation obtains when the integrals in (7) and (10) vanish. A much better approximation for low flows is obtained, however, from (7) and (11).

We take as a first approximation*

$$P(t_1 \mid t_1) \sim P_1(t_1 \mid y_1) = 0 \qquad \text{for} \qquad y_1 < 0, \tag{15a}$$

i.e. there is zero probability that a car entering in one cycle will leave in a later cycle. Substitution of this into (11) then gives as a first approximation for $y_1 > 0$

$$\begin{aligned} P_1(t_1 \mid y_1) &= \int_{y_1}^{\infty} dy\, f(y) = 1 - \int_0^{y_1} dy\, f(y) \qquad &\text{for} \qquad &0 < y_1 < T^* \\ &= 0 & &y_1 > T^*. \end{aligned} \tag{15b}$$

This first approximation corresponds to the statement that a car leaves in a position t_1 if it arrives during the corresponding red period and if it was also the first car to arrive in this period.

One can make a sequence of approximations based upon this first approximation by successive substitution of each new approximation back into the equations to obtain the next approximation. Each new approximation will give a correction due to the influence of cars arriving in cycles further and further removed from the reference car.

If one expresses all the P's in terms of the solution of (14), it suffices to apply this procedure simply to this equation. Thus we obtain for $y_1 < 0$

$$\begin{aligned} P_1(t_1 \mid y_1) &= 0, \\ P_2(t_1 \mid y_1) &= \int_0^{T^*} dy\, f^{(n)}(y_1 + T - y) \int_y^{\infty} dy'\, f(y'), \\ P_j(t_1 \mid y_1) &= \int_0^{\infty} dy\, P_{j-1}(t_1 \mid -y) \Big\{ f^{(n)}(y + y_1 + T) \\ &\quad - \int_0^{T^*} dy'\, f^{(n)}(y_1 + T - y') \times \sum_{l=1}^{\infty} f^{(l)}(y + y', y') \Big\} \qquad \text{for} \qquad j > 2, \end{aligned} \tag{16}$$

and

$$P(t_1 \mid y_1) = \sum_{j=1}^{\infty} P_j(t_1 \mid y_1). \tag{17}$$

Successive corrections generally become increasingly more difficult to evaluate and so the value of this procedure will be limited in practice to situations in which the first few approximations already give results of sufficient accuracy.

4. Ordered flow. To illustrate the above scheme, we consider two extreme cases. The simplest case is that of a completely ordered arrangement of incoming cars, uniformly spaced in time; to be considered in this section. The second case, to be considered in the next section represents the opposite extreme, namely that with the maximum disorder possible for this model.

In the former case, d is chosen to be the time interval between consecutive equally spaced cars and $f(y)$ is chosen to be

$$\begin{aligned} f(y) &= \delta(y - d), \qquad \delta = \text{Dirac } \delta\text{-function} \\ f^{(l)}(y) &= \delta(y - ld). \end{aligned} \tag{18}$$

*The subscripts on P will be used here to denote successive corrections in this approximation scheme and are not to be confused with those used in Sec. 1 (and later discarded).

It follows immediately from (16) that if $nd > T$, i.e. if the incoming flow is less than the critical flow, then $P_2(t_1 \mid y_1) = 0$ for $y_1 < 0$ and consequently all higher corrections also vanish.

$P(t_1 \mid y_1)$ is given exactly from (11) by

$$P(t_1 \mid y_1) = \begin{cases} 1 & \text{if} \quad 0 < y_1 < d \text{ and } T^* \\ 0 & \text{otherwise.} \end{cases} \tag{19a}$$

From (5) one then finds

$$\begin{aligned} P(t_l \mid y_1) &= \begin{cases} 1 & \text{if} \quad (l-1)d < y_1 < ld \quad \text{and} \quad y_1 < t_l \\ 0 & \text{otherwise} \quad \text{for} \quad d < T^*, \end{cases} \\ &= \begin{cases} 1 & \text{if} \quad (l-1)d < y_1 < (l-1)d + T^* \quad \text{and} \quad y_1 < t_l \\ 0 & \text{otherwise} \quad d > T^*. \end{cases} \end{aligned} \tag{19b}$$

Thus the complete solution is in this case quite elementary. We can pursue the analysis further and investigate the distribution of delay times. We define $p(t)$ such that $p(t)dt$ is the probability of a delay of t to t plus dt. With the uniform distribution of arrival time for the reference car, in accordance with earlier postulates, one finds that

$$p(t) = \frac{1}{T} \sum_{l=1}^{n} P(t_l \mid t_l - t). \tag{20}$$

There will always be a non-zero probability for no delay. $p(t)$ is therefore defined as above only for $t > 0$. The probability of zero delay is related to $P(\tau \mid \tau)$ which was eliminated from the integral equations by the normalization condition. The probability of zero delay is obtained also from a normalization condition and is given by

$$1 - \int_0^\infty p(t)\, dt.$$

Substitution of (19) into (20) gives first of all

$$p(t) = 0 \qquad \text{for} \qquad t < 0 \qquad \text{or} \qquad t > t_1 .$$

For $0 < t < t_1$, there are numerous special cases depending upon the relative size of d, T^* and t, but in any case $p(t)$ usually has, for various t, values of 0, 1 or 2 depending upon whether 0, 1 or 2 of the step functions in (19) overlap for that t when substituted into (20). In any particular case it is a simple task to find the appropriate explicit form. In a typical case $p(t)$ will oscillate between the values 1 and 2 in this interval as illustrated in Fig. 3 for a particular choice of parameters.

From $p(t)$ one can calculate averages of any function of t. In particular, one can evaluate the average delay

$$\langle t \rangle = \frac{1}{T} \sum_{l=1}^{\infty} \int_0^\infty t\, dt\, P(t_l \mid t_l - t). \tag{21}$$

For any given set of constants for the traffic light itself, one can easily calculate $\langle t \rangle$ as a function of d or as a function of $\beta \equiv T/nd$, the ratio of the flow rate to the critical rate. Figure 4 shows the results of an exact calculation for a particular choice of T^*, δ and t_1. Although the graph appears on this scale to be quite smooth, it really has a

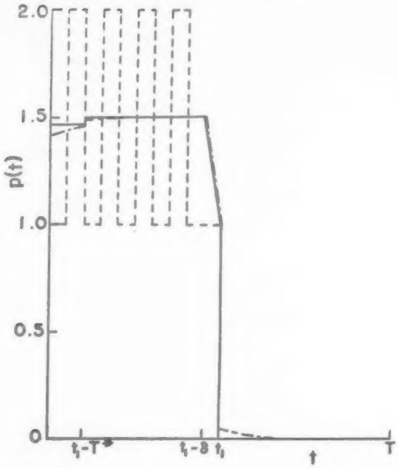

FIG. 3. Distribution of delay times $p(t)$ plotted vs. t for a traffic light with $T = 1$, $T^* = 0.4$, $\delta = 1/20$, $t_1 = 0.5$, $n = 10$ and a flow parameter $\beta = 2/3$ ($\alpha = 10$, $d = 1/20$).
- - - - - ordered flow
———— disordered flow, first approximation
— - — - disordered flow, second approximation.

large number of discontinuities in the derivatives resulting from the numerous discontinuities in the function $p(t)$.

As a practical procedure for obtaining reasonable estimates of the average delay for equally spaced cars, the above method gives much more detail and requires much more work than is justified by the crudeness of the model. In most cases, particularly when n is not a small integer, the above formulation will give results differing only slightly from the formulas for $\langle t \rangle$ given by Clayton [3] on the basis of somewhat simpler assumptions. Clayton calculates $\langle t \rangle$ on the basis of a completely specified sequence of arrival times whereas it has been assumed here that any given car arrives at a randomly distributed time but that the differences in arrival times of consecutive cars are specified.

In terms of the notation of this paper, Clayton's formula reads

$$\langle t \rangle = \frac{(t_1 - \delta/2)^2}{2T(1 - n\,\delta\beta/T)}. \tag{22}$$

The above formula is also plotted in Fig. 4 for comparison, using the same parameters. It can best be described perhaps as a smooth out version of the previous results. The differences between the two curves would however be more accentuated if we had chosen to compare them for a light described by a smaller value of n or one for which δ was not so small compared with t_1 .

5. Disordered flow. The postulates made in Sec. 1 exclude the possibility of applying the above procedure directly to a flow which is completely disordered, i.e. one for which the spacing of incoming cars obeys a Poisson distribution. Although this type of flow has been investigated by several people, it is not obvious that it is representative of the true situation especially for high density of cars near critical flow, the only circumstance in which the statistical distribution of cars may have appreciable consequences.

The assumption has been made in Sec. 1 that cars cannot arrive at time intervals

less than δ. This assumption has been used at several places in the derivation and, at least for certain single lane highways, is perhaps a reasonable one even for high density flows.

Instead of considering a Poisson distribution which corresponds to a distribution of arrival times with maximum disorder consistent with a known value of the average flow or average time interval between cars, we shall consider here a distribution with maximum disorder consistent with the minimum separation assumption and a known value of the average flow or time interval between cars.

The distribution satisfying these specifications must be of the form

$$\begin{aligned} f(y) &= \alpha e^{-\alpha(t-\delta)} \qquad &&\text{for} \quad y > \delta \\ &= 0 &&\text{for} \quad y < \delta \end{aligned} \tag{23}$$

in which α is to be found from the known value of the average of y, denoted again by d as in the previous case

$$d = \delta + 1/\alpha, \qquad \alpha = (d - \delta)^{-1}; \tag{24}$$

$f(y)$ is essentially a Poisson distribution with a displaced origin.

One easily calculates from (6) that

$$\begin{aligned} f^{(j)}(y) &= \alpha^j \frac{e^{-\alpha(y-j\delta)}}{(j-1)!}(y - j\delta)^{j-1} \qquad &&\text{for} \quad y > j\delta \\ &= 0 &&\text{for} \quad y < j\delta \end{aligned} \tag{25}$$

and from (12)

$$f^{(j)}(y, y_1) = \begin{cases} f^{(j)}(y) & \text{for} \quad y_1 < \delta, j > 1 \\ e^{-\alpha(y_1-\delta)} f^{(j)}(y - y_1 + \delta) & \text{for} \quad y_1 > \delta, j > 1. \end{cases} \tag{26}$$

The sequence of approximations represented by Eqs. (16) and (17) does not yield a simple exact solution as it did for the ordered flow, but it does give a very rapidly convergent series of approximations except for flows very close to the critical value.

From the first approximation in (16), one obtains

$$P_1(t_1 \mid y_1) = \begin{cases} e^{-\alpha(y_1-\delta)} & \text{for} \quad \delta < y_1 < T^* \\ 1 & \text{for} \quad 0 < y_1 < \delta \\ 0 & \text{for} \quad y_1 < 0 \quad \text{or} \quad y_1 > T^* \end{cases} \tag{27}$$

and for $l \geq 2$, $P_1(t_l \mid y_1)$ can be found in terms of the function*

$$\begin{aligned} I[u, p] &\equiv \frac{1}{\Gamma(p+1)} \int_0^{u(p+1)^{1/2}} v^p e^{-v}\, dv \qquad &&\text{for} \quad u > 0 \\ &\equiv 0 &&\text{for} \quad u < 0. \end{aligned} \tag{28}$$

To calculate $p(t)$, we again use Eq. (20) but it is convenient to modify the approximation scheme slightly. Substitution of (5) into (20) gives

$$p(t) = \frac{1}{T}\left\{P(t_1 \mid t_1 - t) + \sum_{l=2}^{n} \int_0^\infty dy\, P(t_1 \mid t_l - t - y) f^{(l-1)}(y)\right\}.$$

*This function has been tabulated for various values of u and p in *Tables of the incomplete* **Γ-*function*** edited by K. Pearson, London, 1922.

If one substitutes (25) into this, one obtains

$$p(t) = T^{-1}\left\{P(t_1 \mid t_1 - t) + \int_0^\infty \alpha\, dz\, P(t_1 \mid t_1 - t - z)e^{-\alpha z} \sum_{l=2}^{n} \frac{(\alpha z)^{l-2}}{(l-2)!}\right\}.$$

The summation now represents the first $(n - 2)$ terms in the expansion of $e^{\alpha z}$. It is convenient to write

$$e^{-\alpha z} \sum_{l=2}^{n} (\alpha z)^{l-2}/(l-2)! = 1 - I[\alpha z(n-1)^{-1/2}, n-2].$$

If now one calculates a sequence of approximate values of $p(t)$ from the series of approximate values of $P(t_1 \mid y_1)$, the contribution to $p(t)$ arising from $I[\alpha z(n - 1)^{-1/2}, n - 2]$ is usually small in each such approximation and can logically be incorporated at each step into the next higher approximation. Thus we calculate approximate values of $p(t)$ according to the following scheme.

$$p_1(t) = \frac{1}{T}\left\{P_1(t_1 \mid t_1 - t) + \int_0^\infty \alpha\, dz\, P_1(t_1 \mid t_1 - t - z)\right\},$$

$$\begin{aligned} p_j(t) = \frac{1}{T}\Big\{&-\int_0^\infty \alpha\, dz\, P_{j-1}(t_1 \mid t_1 - t - z) I[\alpha z(n-1)^{-1/2}, n-2] \\ &+ P_j(t_1 \mid t_1 - t) + \int_0^\infty \alpha\, dz\, P_j(t_1 \mid t_1 - t - z)\Big\} \quad \text{for} \quad j > 1 \end{aligned} \tag{29}$$

$$p(t) = \sum_{j=1}^{\infty} p_j(t).$$

Substitution of (27) into (29) for the first approximation gives only elementary integrations resulting in the expression

$$p_1(t) = \begin{cases} 0 & \text{for} \quad t_1 < t \quad \text{or} \quad t < 0 \\ T^{-1}\{1 + \alpha(t_1 - t)\} & \text{for} \quad t_1 - \delta < t < t_1 \\ T^{-1}\{1 + \alpha\delta\} & \text{for} \quad t_1 - T^* < t < t_1 - \delta \\ T^{-1}\{1 + \alpha\delta - e^{-\alpha(T^* - \delta)}\} & \text{for} \quad 0 < t < t_1 - T^* \end{cases} \tag{30}$$

This function is plotted in Fig. 3 where it is compared with the corresponding curve for ordered flow.

The evaluation of the first approximation to $\langle t \rangle$ from $p_1(t)$ is also elementary and gives

$$\langle t \rangle_1 = (1 + \alpha\delta)\frac{t_1^2}{2T} - \frac{(t_1 - T^*)^2}{2T} e^{-\alpha(T^* - \delta)} - \frac{\alpha\delta^2}{2T}\left(t_1 - \frac{\delta}{3}\right). \tag{31}$$

In most practical cases, this is also very close to Clayton's formula. If we assume in (31) that δ and $t_1 - T^*$ are both small compared with t_1, (31) gives

$$\langle t \rangle_1 \sim (1 + \alpha\delta)t_1^2/2T$$

and if we again let

$$\beta = \frac{T}{nd} = \frac{T}{n(\alpha^{-1} + \delta)},$$

we obtain

$$\langle t \rangle_1 \sim \frac{t_1^2}{2T(1 - n\delta\beta/T)}.$$

This is the same as Clayton's formula (22), in which the same approximation of neglecting δ as compared with t_1 is made.

Equation (31) is plotted in Fig. 4 as a function of β along with the corresponding results from the last section.

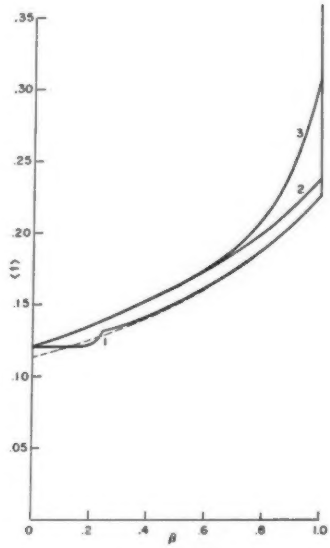

FIG. 4. The average delay $\langle t \rangle$ is plotted as a function of the dimensionless traffic volume β for the same traffic light constants as in Fig. 3. The broken line is Clayton's formula. Solid curve 1, is the curve for ordered flow (Sec. 4). Curve 2 is the first approximation for the disordered flow (Sec. 5) and Curve 3 is the second approximation for disordered flow (Sec. 6).

6. Second approximation—disordered flow. From Eq. (16) and those of the last section, one obtains the following expression for the second approximation $P_2(t_1 \mid y_1)$ for $y_1 < 0$.

$$P_2(t_1 \mid y_1) \begin{cases} = I[\alpha(y_1 + T - n\delta)n^{-1/2}, n - 1] \\ \qquad - I[\alpha(y_1 + T - (n+1)\delta)(n+1)^{-1/2}, n] \\ \qquad\qquad \text{for} \quad y_1 < -T + T^* + n\delta \\ = I[\alpha(y_1 + T - n\delta)n^{-1/2}, n - 1] \\ \qquad - I[\alpha(y_1 + T - (n+1)\delta)(n+1)^{-1/2}, n] \\ \qquad - \exp[-\alpha(y_1 + T - (n+1)\delta)]\alpha^n(y_1 + T - T^* - n\delta)^n/n! \\ \qquad\qquad \text{for} \quad -T + T^* + n\delta < y_1 < 0. \end{cases} \tag{32}$$

The substitution of (32) into the various other formulas will lead to some rather cumbersome expressions. Fortunately, the term causing most of the formal complications, namely the term of (32) with the exponential, can usually be neglected.

If one discards this small term, (32) becomes

$$\begin{aligned} P_2(t_1 \mid y_1) &= I[\alpha(y_1 + T - n\delta)n^{-1/2}, n - 1] \\ &\quad - I[\alpha(y_1 + T - (n+1)\delta)(n+1)^{-1/2}, n] \qquad \text{for} \quad y_1 < 0. \end{aligned} \tag{32a}$$

To obtain $P_2(t_1 \mid y_1)$ for $y_1 > 0$, one must substitute this into (11) and take only the terms not already included in the first approximation (27). Such a calculation gives

$$\begin{aligned} P_2(t_1 \mid y_1) &= -I[\alpha(y_1 + T - (n+1)\delta)(n+1)^{-1/2}, n] && \text{for} \quad 0 < y_1 < \delta, \\ &= -e^{-\alpha(y_1 - \delta)} I[\alpha(T - n\delta)(n+1)^{-1/2}, n] && \text{for} \quad \delta < y_1 < T^*, \\ &= 0 && T^* < y_1 . \end{aligned} \tag{32b}$$

From (32a, b), one can now compute the next approximation to $P(t_l \mid y_1)$ for $l > 1$, and $p_2(t)$, however we shall pass over this somewhat unpleasant and uninteresting phase of the calculation.

The evaluation of $\langle t \rangle_2$, the second order term for the average delay can be evaluated exactly from (33) with no trouble, but we give below only a much simplified approximate expression which is correct to within a fractional error of $(\alpha\delta)^2/2n$ or $\alpha(t_1 - T)^2 \times \exp[-\alpha(T^* - \delta)]/2n$. In the example considered for illustration, the error is actually less than 3% of the second order term alone. This expression is

$$\langle t \rangle_2 \sim \alpha^{-1}(T - n\delta)(1 + \alpha\delta) \int_0^{\alpha(T - n\delta)} dy\, I[yn^{-1/2}, n - 1] \tag{33}$$

which can be evaluated numerically with the aid of the relation

$$\int_0^z dy\, I[yn^{-1/2}, n - 1] = zI[zn^{-1/2}, n - 1] - nI[z(n+1)^{-1/2}, n].$$

$\langle t \rangle_1 + \langle t \rangle_2$ is plotted as Curve 3 in Fig. 4 for comparison with previous results.

7. Conclusions. Even though the analysis so far does not include an estimate of the errors arising from the mathematical approximations (this can be done, however), certain qualitative conclusions are already quite apparent.

Figure 3 shows that delays for the ordered flow, the first approximation to the disordered flow, and Clayton's formula differ by an amount which would in most cases be considered as negligible in view of the crudeness of the model. Clayton's formula, being the simplest to evaluate, still remains the most useful expression at least for low density flows.

It is apparent that the approximation scheme must break down for $\beta \to 1$ and that $\langle t \rangle$ must become infinite. Even the second approximation gives a correction that is quite sensitive to changes in β near $\beta = 1$ and is very small over a considerable range of β, being of order β^n for small β. The third approximation will be even more sensitive to β near $\beta = 1$, will be of order β^{2n} for small β and will be negligible over a larger range of β than the second approximation.

The assumption that the cars approach the light with a certain minimum separation, has a considerable influence on the behavior of $\langle t \rangle$ near $\beta = 1$. As cars become more densely

packed, keeping a certain minimum separation, the uncertainty in the position of the cars decreases at a much faster rate than if cars were allowed to have any separation with the usual Poisson distribution. The delays calculated here will lie somewhere between those one would calculate for the Poisson distribution and those for a completely ordered distribution.

Which type of flow is most representative of the true situation is a difficult question to answer. A completely ordered flow is certainly unrealistic even though it will in many problems, such as that considered here, give simple, qualitatively correct estimates of certain features of the traffic. For low volumes of traffic, the difference between the two possible types of disordered flow is small, but, in any case, as far as delays are concerned, it makes little difference what kind of flow pattern exists.

For high density flows, the degree of disorder does make a difference. Quite likely, the disordered flow considered here is more realistic for single-lane highways and the completely disordered flow more realistic for a multiple-lane highway although the dynamics of the latter situation is not too well defined. A real experimental test of this, however, would not be possible except for nearly critical flows and since the delay is very sensitive to the density, it would be necessary that one have an accurately defined volume of traffic in an equilibrium situation with no transients or disturbances of any kind not specifically taken into consideration here. This is particularly important for high volumes because any slight disturbance of the equilibrium will cause queues to build up very rapidly and dissipate very slowly.

One result of these calculations does seem rather surprising, namely that even to a second approximation, in the example considered, the average delay is less than a third of a cycle. It is also apparent that one will not obtain average delays of more than about one cycle on the basis of this model until the volume of flow has reached a value extremely close to the critical value.

Acknowledgment. The author is indebted to Professor William Prager for instigating this study of traffic problems and for performing himself most of the unpleasant tasks necessary to get such an investigation started.

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2. J. S. Wardrop, *Some theoretical aspects of road traffic Research*, Proc. Inst. Civil Engrs., **II 1**, 325, 1952, and fairly extensive references included in these articles
3. A. J. H. Clayton, *Road traffic calculations*, J. Inst. Civil Engrs. **16**, 247 (1941); see also Wardrop, *loc. cit.*

BOOK REVIEWS

Advanced mathematics for engineers. By H. W. Reddick and F. H. Miller. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1955. xiv + 548 pp. $6.50.

This is the third edition of a very adequate text, first published in 1938 and which presents to undergraduate engineering students a wide variety of advanced mathematical topics and techniques. This edition, prepared by F. H. Miller, is essentially the same as the second edition. Several new sections and topics have been added, but the great bulk of material is unchanged. The problem lists have been extensively revised; some new problems have been added and, for some reason, almost all of the formal drill problems have been trivially altered so that the answers (all of which are given) are different from those of the corresponding problems in the second edition.

PETER CHIARULLI

Lectures on partial differential equations. By I. G. Petrovsky. Interscience Publishers, Inc., New York and London, 1954. x + 245 pp. $5.75.

This book is a translation, by A. Shenitzer, of Petrovsky's concise introduction to the subject of partial differential equations. Chapter I deals with Cauchy's problem, S. Kowalewsky's existence theorem, characteristics, E. Holmgren's uniqueness proof for Cauchy's problem, canonical forms for second order linear partial differential equations in one unknown function of two independent variables, and canonical forms for systems of linear first order partial differential equations in two independent variables. Chapter II is divided into two parts: (a) Cauchy's problem in the domain of non-analytic functions and (b) vibrations. Part (a) deals with the "correct posing" of Cauchy's problem, Cauchy's problem for the wave equation in one, two, and three space dimensions and for hyperbolic systems in two independent variables, Lorentz transformations, mathematical foundations of special relativity. Part (b) is concerned with vibration problems, the so-called "mixed" problems for the wave equation, and specially Fourier's method (expansion in terms of particular solutions obtained by the method of separation of variables) for the vibrating string equation. Chapter III is devoted to elliptic equations, and covers Laplace's equation, potential theory, solution of Dirichlet's problem for a circle by Poisson's integral. The uniqueness of the solution of Dirichlet's problem is proved by an elementary method (not involving Green's theorem) due to I. I. Privalov (Mat. Sbornik (1) 32, 464-469 (1925)) and the existence of the solution by the Poincare-Perron method of sub- and super-harmonic functions. The difference equation method for the approximate solution of the Dirichlet problem is also considered. Parabolic equations are discussed briefly in Chapter IV. At the end of each of the last three chapters there is a brief but informative survey of related known results. To quote from Professor Courant's foreword to this volume: "It will be highly welcome to English speaking students that Petrovsky's masterly lectures on this important subject are now being made accessible through the present translation from the Russian original."

J. B. DIAZ

Fünfstellige Tafeln der Kreis-und Hyperbelfunktionen sowie der Funktionen e^x und e^{-x} mit den natürlichen Zahlen als Argument. By Keiichi Hayashi. Walter de Gruyter & Co., Berlin, 1955. iv + 182 pp. $3.00.

As far as this reviewer can determine, this is a photo-offset reproduction of the 1921 edition of this work, the contents of which are well known and need not be detailed here.

W. PRAGER

(*Continued on p. 392*)

ON THE FLEXURAL VIBRATIONS OF CIRCULAR AND ELLIPTICAL PLATES*

BY

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General Electric Advanced Electronics Center at Cornell University

In this paper we express R. D. Mindlin's version of plate flexure equations, which take transverse shear and rotary inertia into account, in general orthogonal curvilinear coordinates and then we specialize these to polar and elliptical coordinates in order to find the frequency equations for the normal modes of vibration of the circular and elliptical plates respectively. In particular, we wish to discover those of the eight natural boundary conditions for which the normal modes of vibration are expressible in terms of product functions.

In rectangular coordinates, the bending moments (M_x , M_y), twisting moments ($M_{yx} = - M_{xy}$) and shear (Q_x , Q_y) are given by the equations:

$$M_x = D\left(\frac{\partial\psi_x}{\partial x} + \mu\,\frac{\partial\psi_y}{\partial y}\right), \qquad M_y = D\left(\frac{\partial\psi_y}{\partial y} + \mu\,\frac{\partial\psi_x}{\partial x}\right),$$

$$M_{yx} = -M_{xy} = \frac{1-\mu}{2}\,D\left(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial x}\right), \tag{a}$$

$$Q_x = k^2Gh\left(\frac{\partial w}{\partial x} + \psi_x\right), \qquad Q_y = k^2Gh\left(\frac{\partial w}{\partial y} + \psi_y\right),$$

where ψ_x , ψ_y , and w are plate displacements; D, G, and μ are the plate modulus, shear and Poisson's ratio respectively; h is the plate thickness, and $k^2 = \pi^2/12$ is a constant for any plate.

In the case of free vibrations, Mindlin has shown that w, ψ_x , and ψ_y can be expressed in terms of the three functions w_1 , w_2 , and w_3 by the following equations:

$$w = w_1 + w_2\,,$$

$$\psi_x = (\sigma_1 - 1)\,\frac{\partial w_1}{\partial x} + (\sigma_2 - 1)\,\frac{\partial w_2}{\partial x} + \frac{\partial w_3}{\partial y}\,, \tag{b}$$

$$\psi_y = (\sigma_1 - 1)\,\frac{\partial w}{\partial y} + (\sigma_2 - 1)\,\frac{\partial w_2}{\partial y} - \frac{\partial w_3}{\partial x}\,,$$

where w_1 and w_2 are components of the displacement perpendicular to the middle plane of the plate, and w_3 is the potential function which gives rise to the twist about the normal to the plane of the plate;

$$\sigma_1 = \delta_1^2(S^{-1} + R\delta_0^4)^{-1}, \qquad \sigma_2 = \delta_2^2(S^{-1} + R\delta_0^4)^{-1},$$

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where $R = \dfrac{h^2}{12}$ (coefficient of rotary inertia),

$S = \dfrac{D}{k^2Gh}$ (coefficient of transverse shear),

$\delta_0^4 = \dfrac{\rho p^2 h}{D}$ where ρ and p are the plate density and angular frequency respectively;

$$\delta_1^2 = \tfrac{1}{2}\delta_0^2\{(R + S) + [(R - S)^2 + 4\delta_0^{-4}]^{1/2}\},$$

and

$$\delta_2^2 = \tfrac{1}{2}\delta_0^2\{(R + S) - [(R - S)^2 + 4\delta_0^{-4}]^{1/2}\}.$$

Mindlin showed further that the w_i are governed by the following three separated wave equations

$$(\nabla^2 + \delta_i^2)w_i = 0, \qquad i = 1, 2, 3, \tag{c}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and} \qquad \delta_3^2 = \frac{2(R\delta_0^4 - S^{-1})}{1 - \mu}.$$

The functions w_i are also linked through the following boundary conditions: One member of each of the following three products must be specified on the boundary:

$$\psi_\xi M_\xi, \qquad \psi_\eta M_{\xi\eta}, \qquad wQ_\xi,$$

where

$$w = w_1 + w_2, \qquad \psi_\xi = \psi_x \cos\theta + \psi_y \sin\theta, \qquad \psi_\eta = \psi_y \cos\theta - \psi_x \sin\theta,$$

$$Q_\xi = Q_x \cos\theta + Q_y \sin\theta, \qquad M_\xi = M_x \cos^2\theta + M_y \sin^2\theta + 2M_{yx} \sin\theta\cos\theta, \tag{d}$$

$$M_{\xi\eta} = (M_y - M_x)\sin\theta\cos\theta + M_{yx}(\cos^2\theta - \sin^2\theta),$$

θ being the angle between the normal to the boundary and the x-axis.

The classical Lagrange theory of plates is a good approximation only when the wave length is large in comparison with the thickness of the plate, and this restricts the theory generally to low frequency vibrations. The present theory permits extensions to moderately high frequency modes, essentially because it includes coupling between flexual and shear motions.

Transforming equations (*a*), (*b*), and (c) into general orthogonal curvilinear coordinates we have:

$$w = w_1 + w_2,$$

$$\psi_\xi = (\sigma_1 - 1)h_1 \frac{\partial w_1}{\partial \xi} + (\sigma_2 - 1)h_1 \frac{\partial w_2}{\partial \xi} + h_2 \frac{\partial w_3}{\partial \eta},$$

$$\psi_\eta = (\sigma_1 - 1)h_2 \frac{\partial w_1}{\partial \eta} + (\sigma_2 - 1)h_2 \frac{\partial w_2}{\partial \eta} - h_1 \frac{\partial w_3}{\partial \xi},$$

$$Q_\xi = k^2Gh\left[\sigma_1 h_1 \frac{\partial w_1}{\partial \xi} + \sigma_2 h_1 \frac{\partial w_2}{\partial \xi} + h_2 \frac{\partial w_3}{\partial \eta}\right],$$

$$M = D\Big((\sigma_1 - 1)\Big\{h_1^2 \frac{\partial^2 w_1}{\partial_\xi^2} + \mu h_2^2 \frac{\partial^2 w_1}{\partial \eta^2} + P_{1\xi}(x, y) \frac{\partial w_1}{\partial \xi} + P_{2\eta}(x, y) \frac{\partial w_1}{\partial \eta}\Big\}$$

$$+ (\sigma_2 - 1)\Big\{h_1^2 \frac{\partial^2 w_2}{\partial_\xi^2} + \mu h_2^2 \frac{\partial^2 w_2}{\partial \eta^2} + P_1(x, y) \frac{\partial w_2}{p_\xi} + P_{2\eta}(x, y) \frac{\partial w_2}{\partial \eta}\Big\} \tag{A}$$

$$- (1 - \mu)\Big[h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 - \Big(\frac{\partial y}{\partial \xi}\Big)^2\Big]\Big\} \frac{\partial^2 w_3}{\partial \xi\, \partial \eta}$$

$$+ R_{1\xi}(x, y) \frac{\partial w_3}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_3}{\partial \eta}\Big]\Big),$$

$$M_{\xi i} = (1 - \mu)D\Big[(\sigma_1 - 1)\Big(h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2$$

$$- \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big]\Big\} \frac{\partial^2 w_1}{\partial \xi\, \partial \eta} + R_{1\xi}(x, y) \frac{\partial w_1}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_1}{\partial \eta}\Big)$$

$$+ (\sigma_2 - 1)\Big(h_1^4 h_2^2\Big\{4 \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \Big(\frac{\partial y}{\partial \xi}\Big)^2 + \Big(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}\Big)\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 - \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big]\Big\} \frac{\partial^2 w_2}{\partial \xi\, \partial \eta}$$

$$+ R_{1\xi}(x, y) \frac{\partial w_2}{\partial \xi} + R_{2\eta}(x, y) \frac{\partial w_2}{\partial \eta}\Big)$$

$$- \frac{1}{2}\Big(h_1^2 \frac{\partial^2 w_3}{\partial \xi^2} - h_2^2 \frac{\partial^2 w_3}{\partial \eta^2} + S_1(x, y) \frac{\partial w_3}{\partial \xi} + S_{2\eta}(x, y) \frac{\partial w_3}{\partial \eta}\Big)\Big],$$

$$h_1^2 \frac{\partial^2 w_i}{\partial \xi^2} + h_2^2 \frac{\partial^2 w_i}{\partial \eta^2} + L_{1\xi}(x, y) \frac{\partial w_i}{\partial \xi} + L_{2\eta}(x, y) \frac{\partial w_i}{\partial \eta} = 0, \qquad i = 1, 2, 3, \tag{B}$$

where:

$$L_{ij}(x, y) \equiv \Big[h_1^2 \frac{\partial x}{\partial_\xi} \frac{\partial}{\partial_\xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big) + h_2^2 \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)\Big]$$

$$+ \Big[h_i^2 \frac{\partial y}{\partial_\xi} \frac{\partial}{\partial_\xi}\Big(h_i^2 \frac{\partial y}{\partial j}\Big) + h_2^2 \frac{\partial y}{\partial \eta} \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big],$$

$$P_{ij}(x, y) \equiv h_1^4\Big\{\frac{\partial x}{\partial_\xi}\Big[\Big(\frac{\partial x}{\partial \xi}\Big)^2 + 2\Big(\frac{\partial y}{\partial \xi}\Big)^2 - \mu\Big(\frac{\partial y}{\partial \xi}\Big)^2\Big] \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)$$

$$+ \frac{\partial y}{\partial \xi}\Big[\Big(\frac{\partial y}{\partial \xi}\Big)^2 + \mu\Big(\frac{\partial x}{\partial \xi}\Big)^2\Big] \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial y}{\partial_i}\Big)\Big\}$$

$$+ h_2^4\Big\{\frac{\partial x}{\partial \eta}\Big[\mu\Big(\frac{\partial x}{\partial \eta}\Big)^2 + 2\mu\Big(\frac{\partial y}{\partial \eta}\Big)^2 - \Big(\frac{\partial y}{\partial \eta}\Big)^2\Big] \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)$$

$$+ \frac{\partial y}{\partial \eta}\Big[\eta\Big(\frac{\partial y}{\partial \eta}\Big)^2 + \Big(\frac{\partial x}{\partial \eta}\Big)^2\Big] \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big\}. \tag{C}$$

$$R_{ij}(x, y) \equiv h_1^4\Big(\frac{\partial y}{\partial \xi}\Big)^2\Big[\frac{\partial x}{\partial \xi} \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial y}{\partial j}\Big) - \frac{\partial y}{\partial \xi} \frac{\partial}{\partial \xi}\Big(h_i^2 \frac{\partial x}{\partial j}\Big)\Big]$$

$$+ h_2^4 \frac{\partial y}{\partial \eta}\Big[\frac{\partial y}{\partial \eta} \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial x}{\partial j}\Big) - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \eta}\Big(h_i^2 \frac{\partial y}{\partial j}\Big)\Big],$$

$$S_{ii}(x, y) \equiv h_1^4\left\{\frac{\partial x}{\partial \xi}\left[\left(\frac{\partial x}{\partial \xi}\right)^2 + 3\left(\frac{\partial y}{\partial \xi}\right)^2\right]\frac{\partial}{\partial \xi}\left(h_i^2 \frac{\partial x}{\partial j}\right) + \frac{\partial y}{\partial \xi}\left[\left(\frac{\partial x}{\partial \xi}\right)^2 - \left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \xi}\left(h_i^2 \frac{\partial y}{\partial_i}\right)\right\}$$

$$- h_2^4\left\{\frac{\partial x}{\partial \eta}\left[\left(\frac{\partial x}{\partial \eta}\right)^2 + 3\left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \eta}\left(h_i^2 \frac{\partial x}{\partial j}\right) - \frac{\partial y}{\partial \eta}\left[\left(\frac{\partial x}{\partial \eta}\right)^2 - \left(\frac{\partial y}{\partial \eta}\right)^2\right]\frac{\partial}{\partial \eta}\left(h_i^2 \frac{\partial y}{\partial_i}\right)\right\},$$

$$\frac{1}{h_i^2} = \left(\frac{\partial x}{\partial j}\right)^2 + \left(\frac{\partial y}{\partial j}\right)^2, \qquad i = 1 \text{ when } j = \xi \text{ and } i = 2 \text{ when } j = \eta.$$

Specializing equations (A) and (B) into polar and elliptical coordinates we have: for polar coordinates $h_1 = 1$, $h_2 = 1/r$

$$w = w_1 + w_2\,, \qquad \psi_r = (\sigma_1 - 1)\frac{\partial w_1}{\partial r} + (\sigma_2 - 1)\frac{\partial w_2}{\partial r} + \frac{1}{r}\frac{\partial w_3}{\partial \theta}\,,$$

$$\psi_\theta = \frac{\sigma_1 - 1}{r}\frac{\partial w_1}{\partial \theta} + \frac{\sigma_2 - 1}{r}\frac{\partial w_2}{\partial \theta} - \frac{\partial w_3}{\partial r}\,,$$

$$Q_r = k^2Gh\left[\sigma_1\frac{\partial w_1}{\partial r} + \sigma_2\frac{\partial w_2}{\partial r} + \frac{1}{r}\frac{\partial w_3}{\partial \theta}\right], \tag{D}$$

$$M_r = D\left\{(\sigma_1 - 1)\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_1}{\partial r} + \frac{\mu}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] + (\sigma_2 - 1)\left[\frac{\partial^2 w_2}{\partial r^2} + \frac{\mu}{r}\frac{\partial w_2}{\partial r} + \frac{\mu}{r^2}\frac{\partial w_2}{\partial \theta^2}\right]\right.$$

$$\left. + (1 - \mu)\left[\frac{1}{r}\frac{\partial^2 w_2}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial w_3}{\partial \theta}\right]\right\},$$

$$M_{r\theta} = (1 - \mu)D\left\{(\sigma_1 - 1)\left[\frac{1}{r}\frac{\partial^2 w_1}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_1}{\partial \theta}\right] + (\sigma_2 - 1)\left[\frac{1}{r}\frac{\partial^2 w_2}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_2}{\partial \theta}\right]\right.$$

$$\left. - \frac{1}{2}\left[\frac{\partial^2 w_3}{\partial r^2} - \frac{1}{r}\frac{\partial w_3}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_3}{\partial \theta^2}\right]\right\},$$

$$\frac{\partial^2 w_i}{\partial r^2} + \frac{1}{r}\frac{\partial w_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_i}{\partial \theta^2} + \delta_i^2 w_i = 0, \qquad i = 1, 2, 3. \tag{E}$$

For elliptical coordinates

$$h_1^2 = h_2^2 = \frac{1}{C^2(\cosh^2 \xi - \cos^2 \eta)} = \frac{2}{C^2(\cosh 2\xi - \cos 2\eta)}\,,$$

$$w = w_1 + w_2\,, \qquad \psi_\xi = h_1\left[(\sigma_1 - 1)\frac{\partial w_1}{\partial \xi} + (\sigma_2 - 1)\frac{\partial w_2}{\partial \xi} + \frac{\partial w_3}{\partial \eta}\right],$$

$$\psi_\eta = h_1\left[(\sigma_1 - 1)\frac{\partial w_1}{\partial \eta} + (\sigma_2 - 1)\frac{\partial w_2}{\partial \eta} - \frac{\partial w_3}{\partial \eta}\right],$$

$$Q_\xi = k^2Ghh_1\left[\sigma_1\frac{\partial w_1}{\partial \xi} + \sigma_2\frac{\partial w_2}{\partial \xi} + \frac{\partial w_3}{\partial \eta}\right],$$

$$M_\xi = Dh_1^2\left[(\sigma_1 - 1)\left\{\frac{\partial w_1^2}{\partial_\xi^2} + \mu\frac{\partial w_1^2}{\partial_\eta^2} - \frac{C^2h_1^2}{2}(1 - \mu)\left[\sinh 2\xi\frac{\partial w_1}{\partial \xi} - \sin 2\frac{\partial w_1}{\partial \eta}\right]\right\}\right.$$

$$+ (\sigma_2 - 1)\left\{\frac{\partial^2 w_2}{\partial \xi^2} + \mu\frac{\partial^2 w_2}{\partial_\eta{}^2} - \frac{C^2h_i^2}{2}(1 - \mu)\left[\sinh 2\xi\frac{\partial w_2}{\partial \xi} - \sin 2\eta\frac{\partial w_2}{\partial \xi}\right]\right\} \tag{F}$$

$$\left. + (1 - \mu)\left\{\frac{\partial^2 w_3}{\partial \xi\,\partial \eta} - \frac{C^2h_1^2}{2}\left[\sin 2\eta\frac{\partial w_3}{\partial \xi} + \sinh 2\xi\frac{\partial w_3}{\partial \eta}\right]\right\}\right],$$

$$M_{\xi\eta} = \frac{1-\mu}{2} Dh_1^2\left\{(\sigma_1 - 1)\left[2\frac{\partial^2 w_1}{\partial\xi\,\partial\eta} - C^2h_1^2\left(\sin 2\eta\,\frac{\partial w_1}{\partial\xi} + \sinh 2\xi\,\frac{\partial w_1}{\partial\eta}\right)\right]\right.$$

$$+ (\sigma_2 - 1)\left[2\frac{\partial^2 w_2}{\partial\xi\,\partial\eta} - C^2h_1^2\left(\sin 2\eta\,\frac{\partial w_2}{\partial\xi} + \sinh 2\xi\,\frac{\partial w_2}{\partial\eta}\right)\right]$$

$$\left.- \left[\frac{\partial^2 w_3}{\partial\xi^2} - \frac{\partial^2 w_3}{\partial\eta^2} - C^2h_1^2\left(\sinh 2\xi\,\frac{\partial w_3}{\partial\xi} - \sin 2\eta\,\frac{\partial w_3}{\partial}\right)\right]\right\},$$

$$\frac{\partial^2 w_i}{\partial\xi^2} + \frac{\partial^2 w_i}{\partial\eta^2} + 2k_i^2(\cosh 2\xi - \cos 2\eta)w_i = 0, \qquad i = 1, 2, 3, \tag{G}$$

where $2k_i = \delta_i C$, C is the semi-focal length of the elliptical plate.

We will now find the frequency equations for the normal modes of vibrations of the circular and elliptical plates which satisfy the following eight boundary conditions:

CIRCULAR PLATE		ELLIPTICAL PLATE	
(1) $\psi_r = \psi_\theta = w = 0$	(clamped plate)	$\psi_\xi = \psi_\eta = w = 0$	
(2) $\psi_r = \psi_\theta = Q_r = 0$		$\psi_\xi = \psi_\eta = Q_\xi = 0$	
(3) $\psi_r = M_{r\theta} = w = 0$		$\psi_\xi = M_{\xi\eta} = w = 0$	
(4) $\psi_r = M_{r\theta} = Q_r = 0$		$\psi_\xi = M_{\xi\eta} = Q_\xi = 0$	(H)
(5) $M_r = M_{r\theta} = Q_r = 0$	(free plate)	$M_\xi = M_{\xi\eta} = Q_\xi = 0$	
(6) $M_r = M_{r\theta} = w = 0$		$M_\xi = M_{\xi\eta} = w = 0$	
(7) $M_r = \psi_\theta = Q_r = 0$		$M_\xi = \psi_\eta = Q_\xi = 0$	
(8) $M_r = \psi_\theta = w = 0$		$M_\xi = \psi_\eta = w = 0$	

when $r = r_0$ and $\xi = \xi_0$ respectively.

It should be remarked that the boundary conditions that we are assuming are particular and that other values could be assumed for the quantities involved if desired.

By assuming that the solutions of equations (*E*) and (*G*) are expressible as product solutions we obtain the following respective pairs of ordinary differential equations:

$$\left\{\begin{aligned} &\frac{d^2w_i(r)}{dr^2} + \frac{1}{r}\frac{dw_i(r)}{dr} + \left(\delta_i^2 - \frac{m^2}{r^2}\right)w_i(r) = 0 \\ &\frac{d^2w_i(\theta)}{d\theta^2} + m^2w_i(\theta) = 0 \qquad (i = 1, 2, 3) \end{aligned}\right\} \tag{I}$$

$$\left\{\begin{aligned} &\frac{d^2w_i(\eta)}{d\eta^2} + (a - 2q_i\cos 2\eta)w_i(\eta) = 0 \\ &\frac{d^2w_i(\xi)}{d\xi^2} - (a - 2q_i\cosh 2\xi)w_i(\xi) = 0 \qquad (i = 1, 2, 3) \end{aligned}\right\}, \tag{J}$$

where $q_i = k_i^2$ and a and m^2 are separation constants. The first of equations (I) is Bessel's equation. The first of equations (J) is Mathieu's equation and the second is Mathieu's modified equation. The solutions of (I) and (J) are of the respective forms:

$$w_i(r) = J_m(\delta_i , r), \qquad w_i(\theta) = \begin{pmatrix} \sin m\theta \\ \\ \cos m\theta \end{pmatrix} \qquad i = 1, 2, 3, \tag{K}$$

$$w_i(\xi) = \begin{pmatrix} \mathrm{Ce}_m(\xi, q_i), a = a_m \\ \\ \mathrm{Se}_m(\xi, q_i), b = b_m \end{pmatrix}, \qquad w_i(\eta) = \begin{pmatrix} \mathrm{ce}_m(\eta, q_i), a = a_m \\ \\ \mathrm{se}_m(\eta, q_i), b = b_m \end{pmatrix}, \tag{L}$$

$$i = 1, 2, 3$$

where a_m and b_m are characteristic numbers of the Mathieu functions.

Hence the solutions of (E) and (G) are products of the solutions in (K) and (L) respectively.

We shall assume the following solutions for (E) and (G) throughout in the solution of our two problems:

$$\left.\begin{aligned} w_i^{(m)}(r, \theta) &= A_m^{(i)} J_m(\delta_i , r) \cos m\theta, \qquad i = 1, 2, \\ w_i^{(m)}(r, \theta) &= A_m^{(i)} J_m(\delta_i , r) \sin m\theta, \qquad i = 3, \end{aligned}\right\} \tag{M}$$

$$\left.\begin{aligned} w_i^{(m)}(\xi, \eta) &= C_m^{(i)} \mathrm{Ce}_{2n+1}(\xi, q_i) \mathrm{ce}_{2n+1}(\eta, q_i), \qquad (a = a_m = a_{2n+1}), \\ w_i^{(m)}(\xi, \eta) &= C_m^{(i)} \mathrm{Ce}_{2n+1}(\xi, q_i) \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(q_i) \cos (2r + 1)\eta \qquad \text{for } i = 1, 2, \\ w_i^{(m)}(\xi, \eta) &= C_m^{(i)} \mathrm{Se}_{2n+1}(\xi, q_i) \mathrm{se}_{2n+1}(\eta, q_i), \qquad (b = b_m = b_{2n+1}), \\ w_i^{(m)}(\xi, \eta) &= C_m^{(i)} \mathrm{Se}_{2n+1}(\xi, q_i) \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(q_i) \sin (2r + 1)\eta \qquad \text{for } i = 3. \end{aligned}\right\} \tag{N}$$

By substituting the assumed solutions (M) in the equations (D), making use of the boundary conditions (H), we obtain, after some reductions, the following results for the circular plate:

PROBLEM 1: $\psi_r = \psi_\theta = w = 0$ for $r = r_0$ (clamped plated).

$$\left.\begin{aligned} &A_m^{(1)} J_m(\delta_1 , r_0) + A_m^{(2)} J_m(\delta_2 , r_0) + 0 = 0, \\ &A_m^{(1)}(\sigma_1 - 1) J_m'(\delta_1 , r_0) + A_m^{(2)}(\sigma_2 - 1) J_m'(\delta_2 , r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3 , r_0) = 0, \\ &A_m^{(1)}(\sigma_1 - 1) \frac{m}{r_0} J_m(\delta_1 , r_0) + A_m^{(2)}(\sigma_2 - 1) \frac{m}{r_0} J_m(\delta_2 , r_0) + A_m^{(3)} J_m'(\delta_3 , r_0) = 0. \end{aligned}\right. \tag{1}$$

PROBLEM 2: $\psi_r = \psi_\theta = Q_r = 0$ for $r = r_0$.

$$\left.\begin{aligned} &A_m^{(1)}(\sigma_1 - 1) J_m'(\delta_1 , r_0) + A_m^{(2)}(\sigma_2 - 1) J_m'(\delta_2 , r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3 , r_0) = 0, \\ &A_m^{(1)}(\sigma_1 - 1) \frac{m}{r_0} J_m(\delta_1 , r_0) + A_m^{(2)}(\sigma_2 - 1) \frac{m}{r_0} J_m(\delta_2 , r_0) + A_m^{(3)} J_m'(\delta_3 , r_0) = 0, \\ &A_m^{(1)} \sigma_1 J_m'(\delta_1 , r_0) + A_m^{(2)} \sigma_2 J_m'(\delta_2 , r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3 , r_0) = 0. \end{aligned}\right. \tag{2}$$

PROBLEM 3: $\psi_r = M_{r\theta} = w = 0$ for $r = r_0$.

$$A_m^{(1)} J_m(\delta_1, r_0) + A_m^{(2)} J_m(\delta_2, r_0) + 0 = 0,$$

$$A_m^{(2)}(\sigma_1 - 1)J'_m(\delta_1, r_0) + A_m^{(2)}(\sigma_1 - 1)J'_m(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0.$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J'_m(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J'_m(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + A_m^{(3)} \frac{1}{2}\left[J''_m(\delta_3, r_0) - \frac{1}{r_0} J'_m(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned} \tag{3}$$

PROBLEM 4: $\psi_r = M_r = Q_{r\theta} = 0$ for $r = r_0$.

$$A_m^{(1)}\sigma_1 J'_m(\delta_1, r_0) + A_m^{(2)}\sigma_2 J'_m(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0,$$

$$A_m^{(1)}(\sigma_1 - 1)J'_m(\delta_1, r_0) + A_m^{(2)}(\sigma_2 - 1)J'_m(\delta_2, r_0) + A_m^{(3)} \frac{m}{r_0} J_m(\delta_3, r_0) = 0,$$

$$\begin{aligned} A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J'_m(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J'_m(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + \frac{A_m^{(3)}}{2}\left[J''_m(\delta_3, r_0) - \frac{1}{r_0} J_m(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned} \tag{4}$$

PROBLEM 5: $M_r = M_{r\theta} = Q_r = 0$ for $r = r_0$ (free plate).

$$\begin{aligned} A_m^{(1)}\sigma_1 J'_m(\delta_1, r_0) + A_m^{(2)}\sigma_2 J'_m(\delta_2, r_0) + 0 &= 0, \\ A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J'_m(\delta_1, r_0) - \frac{m}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J'_m(\delta_2, r_0) - \frac{m}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + \frac{A_m^{(3)}}{2}\left[J''_m(\delta_3, r_0) - \frac{1}{r_0} J'_m(\delta_3, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3, r_0)\right] &= 0, \\ A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1, r_0) + \frac{\mu}{r_0} J'_m(\delta_1, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_1, r_0)\right] & \\ + A_m^{(2)}(\sigma_2 - 1)\left[J''_m(\delta_2, r_0) + \frac{\mu}{r_0} J'_m(\delta_2, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_2, r_0)\right] & \\ + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0} J'_m(\delta_3, r_0) - \frac{m}{r_0^2} J_m(\delta_3, r_0)\right] &= 0. \end{aligned} \tag{5}$$

PROBLEM 6: $M_r = M_{r\theta} = w = 0$ for $r = r_0$.

$$A_m^{(1)} J_m(\delta_1\,, r_0) + A_m^{(2)} J_m(\delta_2\,, r_0) + 0 = 0,$$

$$\begin{aligned} &A_m^{(1)}(\sigma_1 - 1)\left[\frac{m}{r_0} J'_m(\delta_1\,, r_0) - \frac{m}{r_0^2} J_m(\delta_1\,, r_0)\right] \\ &\qquad + A_m^{(2)}(\sigma_2 - 1)\left[\frac{m}{r_0} J'_m(\delta_2\,, r_0) - \frac{m}{r_0^2} J_m(\delta_2\,, r_0)\right] \\ &\qquad + \frac{A_m^{(3)}}{2} J''_m(\delta_3\,, r_0) - \frac{1}{r_0} J'_m(\delta_3\,, r_0) + \frac{m^2}{r_0^2} J_m(\delta_3\,, r_0)\Big] = 0, \\ &A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_1\,, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_1\,, r_0)\right] \\ &\qquad + A_m^{(2)}(\sigma_2 - 1)\left[J''_m(\delta_2\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_2\,, r_0) - \frac{\mu m^2}{r_0^2} J_m\; \delta_2\,, r_0)\right] \\ &\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0} J'_m(\delta_3\,, r_0) - \frac{m}{r_0^2} J_m(\delta_3\,, r_0)\right] = 0. \end{aligned} \tag{6}$$

PROBLEM 7: $M_r = \psi_\theta = Q_r = 0$ for $r = r_0$.

$$A_m^{(1)}\sigma_1 J'_m(\delta_1\,, r_0) + A_m^{(2)}\sigma_2 J'_m(\delta_2\,, r_0) + A_m \frac{m}{r_0} J_m(\delta_3\,, r_0) = 0,$$

$$A_m^{(1)}(\sigma_1 - 1)\frac{m}{r_0} J_m(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1)\frac{m}{r_0} J_m(\delta_2\,, r_0) + A_m^{(3)} J'_m(\delta_3\,, r_0) = 0,$$

$$\begin{aligned} &A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_1\,, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_1\,, r_0)\right] \\ &\qquad + A_m^{(2)}(\sigma_2 - 1)\left[J''_m(\delta_2\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_2\,, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_2\,, r_0)\right] \\ &\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0} J'_m(\delta_3\,, r_0) - \frac{m}{r_0^2} J_m(\delta_3\,, r_0)\right] = 0. \end{aligned} \tag{7}$$

PROBLEM 8: $M_r = \psi_\theta = w = 0$ for $r = r_0$.

$$A_m^{(1)} J_m(\delta_1\,, r_0) + A_m^{(2)} J_m(\delta_2\,, r_0) + 0 = 0,$$

$$A_m^{(1)}(\sigma_1 - 1)\frac{m}{r_0} J_m(\delta_1\,, r_0) + A_m^{(2)}(\sigma_2 - 1)\frac{m}{r_0} J_m(\delta_2\,, r_0) + A_m^{(3)} J'_m(\delta_3\,, r_0) = 0,$$

$$\begin{aligned} &A_m^{(1)}(\sigma_1 - 1)\left[J''_m(\delta_1\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_1\,, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_1\,, r_0)\right] \\ &\qquad + A_m^{(2)}(\sigma_1 - 1)\left[J''_m(\delta_2\,, r_0) + \frac{\mu}{r_0} J'_m(\delta_1\,, r_0) - \frac{\mu m^2}{r_0^2} J_m(\delta_2\,, r_0)\right] \\ &\qquad + A_m^{(3)}(1 - \mu)\left[\frac{m}{r_0} J'_m(\delta_3\,, r_0) - \frac{m}{r_0^2} J_m(\delta_3\,, r_0)\right] = 0. \end{aligned} \tag{8}$$

By eliminating the constants from equations (1)-(8) we obtain the frequency equations for each of the problems.

By substituting the assumed solution (N) into equations (F), making use of the boundary conditions (H) and simplifying we obtain the following results for the elliptical plate:

PROBLEM I: $\psi_\xi = \psi_\eta = w = 0$ for $\xi = \xi_0$ (clamped plate).

$$\begin{aligned}
&A_r^{(1)}\mathrm{Ce}_{2n+1}(\xi_0, q_1) + A_r^{(2)}\mathrm{Ce}_{2n+1}(\xi_0, q_2) + 0 = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_2)\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0, q_3) = 0 \qquad\qquad \text{(I)}\\
&A_r^{(1)}(\sigma_1 - 1)(2r+1)\mathrm{Ce}_{2n+1}(\xi_0, q_1) + A_r^{(2)}(\sigma_2 - 1)(2r+1)\mathrm{Ce}_{2n+1}(\xi_0, q_2)\\
&\qquad\qquad + A_r^{(3)}\mathrm{Se}'_{2n+1}(\xi_0, q_3) = 0
\end{aligned}$$

for every value of r.

PROBLEM II: $\psi_\xi = \psi_\eta = Q_\xi = 0$ for $\xi = \xi_0$.

$$\begin{aligned}
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_2) + A_r^{(3)}\mathrm{Se}_{2n+1}(\xi_0, q_3) = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}_{2n+1}(\xi_0, q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}_{2n+1}(\xi_0, q_2) \qquad\qquad \text{(II)}\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0, q_3) = 0,\\
&A_r^{(1)}\sigma_1\mathrm{Ce}'_{2n+1}(\xi_0, q_1) + A_r^{(2)}\sigma_2\mathrm{Ce}'_{2n+1}(\xi_0, q_2) + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0, q_3) = 0.
\end{aligned}$$

for every value of r.

By eliminating the constants in (I) and (II) we obtain the frequency equations for problems I and II respectively.

PROBLEM III: $\psi_\xi = M_{\xi\eta} = w = 0$ for $\xi = \xi_0$.

$$\begin{aligned}
&A_r^{(1)}\mathrm{Ce}_{2n+1}(\xi_0, q_1) + A_r^{(2)}\mathrm{Ce}_{2n+1}(\xi_0, q_2) + 0 = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_1) + A_r^{(2)}(\sigma_2 - 1)\mathrm{Ce}'_{2n+1}(\xi_0, q_2)\\
&\qquad\qquad + A_r^{(3)}(2r+1)\mathrm{Se}_{2n+1}(\xi_0, q_3) = 0,\\
&A_r^{(1)}(\sigma_1 - 1)\{-2(2r+1)[\mathrm{Ce}'_{2n+1}(\xi_0, q_1) - G\,\mathrm{Ce}_{2n+1}(\xi_0, q_1)]\sin(2r+1)_\eta\\
&\qquad\qquad + 2F\mathrm{Ce}'_{2n+1}(\xi_0, q_1)\cos(2r+1)_\eta\}\\
&\quad + A_r^{(2)}(\sigma_2 - 1)\{-2(2r+1)\mathrm{Ce}'_{2n+1}(\xi_0, q_2) - G\,\mathrm{Ce}_{2n+1}(\xi_0, q_2)]\sin(2r+1)_\eta\\
&\qquad\qquad + 2F\mathrm{Ce}'_{2n+1}(\xi_0, q_2)\cos(2r+1)_\eta\} \qquad\qquad \text{(III)}\\
&\quad - A_r^{(3)}\{[\mathrm{Se}''_{2n+1}(\xi_0, q_3) + (2r+1)^2\mathrm{Se}_{2n+1}(\xi_0, q_3) - 2G\mathrm{Se}'_{2n+1}(\xi_0, q_3)]\sin(2r+1)_\eta\\
&\qquad\qquad - 2(2r+1)F\mathrm{Se}_{2n+1}(\xi_0, q_3)\cos(2r+1)_\eta\} = 0,
\end{aligned}$$

where

$$F \equiv -\frac{C^2h_1^2}{2}\sin 2\eta, \qquad G \equiv \frac{C^2h_1^2}{2}\sinh 2\xi_0, \qquad \text{and}$$

$$h_1^2 = h_2^2 = \frac{1}{C^2(\cosh^2\xi - \cos^2\eta)} = \frac{2}{C^2(\cosh 2\xi - \cos 2\eta)}.$$

But since the above equations must be independent of η we cannot solve this problem. It is found that the same thing is true for the remaining five problems of the elliptical plate.

We can thus sum up our conclusions as follows:

Conclusion I: The problem of finding the frequency equations for the normal modes of vibration for a circular plate under the boundary conditions (H) can be solved in closed form and expressed in terms of Bessel functions by assuming product solutions for Mindlin's equations (E).

Conclusion II: The problem of finding the frequency equations for the normal modes of vibration of the elliptical plate, under the boundary conditions (H) can be solved in closed form and expressed in terms of Mathieu functions by assuming product solutions for Mindlin's equations (G), *ONLY* in the cases when the boundary conditions (H) are independent of the bending and twisting moments M_ξ and $M_{\xi\eta}$ respectively. Thus, the normal modes for the important case of the free elliptical edge do not appear to be expressible as product functions in elliptical coordinates.

STRESSES IN A CIRCULAR CYLINDER HAVING A SPHERICAL CAVITY UNDER TENSION*

BY

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Introduction. The stresses in a large tension member having a spherical cavity have been investigated by several authors, including the present writer.[1-4] In dealing with such a problem the surface of the member is commonly assumed to be infinitely distant from the cavity so that the presence of the surface produces no effect on the stresses in the neighborhood of the cavity. However, this assumption is no longer valid when the surface of the member is at a finite distance from the cavity. The consideration of the effect of the surface near the cavity naturally increases the analytic complication of the problem.

In the present paper a solution for an infinite circular cylinder having a spherical cavity under axial tension will be given. The cavity is assumed to be symmetrically located within the cylinder so that the theory of symmetrical strain for solids of revolution[5] can be applied. The solution is obtained by constructing a stress function which satisfies the boundary conditions on the surface of the cylinder as well as at its ends. The boundary conditions on the surface of the cavity are satisfied by adjusting the coefficients of superposition involved in the solution. Here the stress function is necessarily a biharmonic function.[6] The method of solution is first described. Then, as an illustration, numerical examples are given for two different radii of the cavity. In particular, the maximum stresses in the cylinder are calculated and shown graphically.

Method of solution. Denote as usual the cylindrical and spherical coordinates of a point by (r, θ, z) and (ρ, ϕ, θ) respectively. For convenience, r, z and ρ will be regarded as dimensionless quantities referring to a typical length a. They are connected with each other by

$$z = \rho \cos \phi, \qquad r = \rho \sin \phi. \tag{1}$$

Consider an infinite circular cylinder of radius a having a symmetrically located spherical cavity of radius λa as shown in Fig. 1. The axis of the cylinder will be taken

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[1]R. V. Southwell and H. J. Gough, *Concentration of stress in the neighborhood of a small spherical flaw*, etc., Phil. Mag. 1, 71-97 (1926).

[2]J. N. Goodier, *Concentration of stress around spherical and cylindrical inclusions and flaws*, Trans. ASME 55, 39-44 (1933).

[3]H. Neuber, *Theory of notch stresses*, Edwards Brothers, 1946, English translation, pp. 102-104.

[4]C. B. Ling and K. L. Yang, *On symmetrical strain in solids of revolution in spherical coordinates*, Trans. ASME 73, A367-371 (1951). Note that in Eq. [32] and [34], p. 370, the factor $(2n^2 + 2n - 3 + \nu)$ should be read as $(2n^2 - 1 + \nu)$.

[5]A. E. H. Love, *Mathematical theory of elasticity*, fourth edition, Dover Publications, 1944, pp. 274-277.

[6]A different method of solution in such a case based on Boussinesq's approach is to find two harmonic functions. See footnote [3].

as the axis of z and the center of the cavity as the origin. In the absence of the cavity, a uniform axial tension of T per unit area would be given by the stress function

$$\chi_0 = \frac{Ta^3}{6(1+\sigma)} \{3\sigma r^2 z + (1-2\sigma)z^3\}, \tag{2}$$

where σ is Poisson's ratio. The method of satisfying the boundary conditions when the cavity is present is to construct two sets of biharmonic functions each of which gives no traction on the surface of the cylinder and at the same time gives no stress at z infinity or the ends of the cylinder. These functions are combined linearly and added to χ_0. The

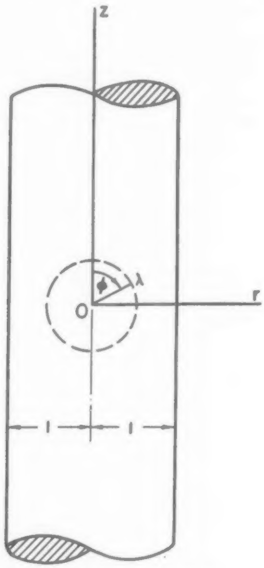

FIG. 1. The cylinder.

boundary conditions on the surface of the cavity are then satisfied by adjusting the coefficients of superposition attached to the functions. The sets of functions are derived by differentiation from two biharmonic functions each of which has a singularity at the origin.

Two sets of biharmonic functions. Consider a biharmonic function as follows:

$$\chi_1 = \frac{a^3}{2} \log \frac{1+\mu}{1-\mu} + a^3 \int_0^\infty \{\psi_1(k) I_0(kr) + \psi_2(k) kr I_1(kr)\} \sin kz \, dk, \tag{3}$$

where I_n are modified Bessel functions of the first kind of order n and ψ_n are arbitrary functions, and

$$\mu = \cos \phi. \tag{4}$$

The first term of this function in fact represents a center of radial tension at the origin. The subsequent integral is added so as to annul the traction on the surface of

the cylinder. This function gives the following normal and tangential stresses at $r = 1$ or the surface of the cylinder.

$$[\sigma_r]_1 = \frac{1}{a^3}\left[\frac{\partial}{\partial z}\left\{\sigma\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) - \frac{\partial^2}{\partial r^2}\right\}\chi_1\right]_{r=1},$$

$$= \frac{1}{(1 + z^2)^{3/2}} - \frac{3}{(1 + z^2)^{5/2}} - \int_0^\infty k^2[\psi_1(k)\{kI_0(k) - I_1(k)\}$$

$$+ \psi_2(k)k\{(1 - 2\sigma)I_0(k) + kI_1(k)\}] \cos kz\, dk,$$

$$[\tau_{rz}]_1 = \frac{1}{a^3}\left[\frac{\partial}{\partial r}\left\{(1 - \sigma)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) - \frac{\partial^2}{\partial z^2}\right\}\chi_1\right]_{r=1},$$

$$= -\frac{3z}{(1 + z^2)^{5/2}} + \int_0^\infty k^3[\psi_1(k)I_1(k) + \psi_2(k)\{kI_0(k) + 2(1 - \sigma)I_1(k)\}] \sin kz\, dk. \quad (5)$$

These stresses are annulled throughout the surface of the cylinder provided that by Fourier transforms,

$$\psi_1(k)\{kI_0(k) - I_1(k)\} + \psi_2(k)k\{(1 - 2\sigma)I_0(k) + kI_1(k)\}$$

$$= \frac{2}{\pi k^2}\int_0^\infty \left\{\frac{1}{(1 + z^2)^{3/2}} - \frac{3}{(1 + z^2)^{5/2}}\right\} \cos kz\, dz,$$

$$\psi_1(k)I_1(k) + \psi_2(k)\{kI_0(k) + 2(1 - \sigma)I_1(k)\} = \frac{2}{\pi k^3}\int_0^\infty \frac{3z \sin kz\, dz}{(1 + z^2)^{5/2}}. \quad (6)$$

In view of the relation[7],

$$K_n(k) = \frac{1\cdot 3\cdot 5 \cdots (2n - 1)}{k^n}\int_0^\infty \frac{\cos kz\, dz}{(1 + z^2)^{(2n+1)/2}}, \quad (7)$$

where K_n are modified Bessel functions of the second kind of order n, we find

$$\begin{aligned}\psi_2(k) &= \frac{2}{\pi}\frac{I_0(k)K_1(k) + I_1(k)K_0(k)}{k^2I_0^2(k) - \{k^2 + 2(1 - \sigma)\}I_1^2(k)},\\ \psi_1(k) &= -\frac{2}{\pi}\frac{kI_0(k)K_0(k) + \{k^2 + 2(1 - \sigma)\}I_1(k)K_1(k)/k}{k^2I_0^2(k) - \{k^2 + 2(1 - \sigma)\}I_1^2(k)} - 2(1 - \sigma)\psi_2(k).\end{aligned} \quad (8)$$

With these values of ψ_1 and ψ_2, it appears that the integral in (3) becomes divergent at the lower limit. However, the divergence can be removed by modifying the integral as follows:

$$\chi_1 = \frac{a^3}{2}\log\frac{1 + \mu}{1 - \mu} + a^3\int_0^\infty [\{\psi_1(k)I_0(kr) + \psi_2(k)krI_1(kr)\} \sin kz + f_1(k)z]\, dk, \quad (9)$$

where

$$f_1(k) = \frac{4}{\pi(1 + \sigma)}\left\{(2 - \sigma)K_0(k) + \frac{3(1 - \sigma)}{k}K_1(k)\right\} \quad (10)$$

[7] G. N. Watson, *Theory of Bessel functions*, 2nd edition, 1944, p. 185, Cambridge University Press. Cf. the integral considered by Poisson and Malmstén. Also see p. 80 for the definition of K_n.

such that the integral becomes convergent at the lower limit. This modification is permissible for it produces no effect on the stresses in the cylinder as well as on the convergence at the upper limit. It can be shown that the function χ_1 thus obtained gives no stress at z infinity or the ends of the cylinder.

It is obvious that successive differentiation with respect to z gives functions with the desired properties on the surface of the cylinder as well as at z infinity, but by symmetry odd derivatives must be excluded since they are even in z and cannot enter into the required solution. The function χ_1 itself may be included since it gives no unbalanced force at the origin. The set of functions is therefore

$$\chi_1\,, \qquad \frac{\partial^2}{\partial z^2}\chi_1\,, \qquad \frac{\partial^4}{\partial z^4}\chi_1\,, \cdots .$$

Again, consider a biharmonic function as follows:

$$\chi_2 = a^3\rho + a^3\int_0^\infty \{\psi_3(k)I_0(kr) + \psi_4(k)krI_1(kr)\}\cos kz\,dk. \tag{11}$$

The first term of this function represents a concentrated force at the origin in the direction of z. This function gives the following normal and tangential stresses at $r = 1$.

$$\begin{aligned}[\sigma_r]_1 = \frac{(1-2\sigma)z}{(1+z^2)^{3/2}} - \frac{3z}{(1+z^2)^{5/2}} + \int_0^\infty k^2[\psi_3(k)\{kI_0(k) - I_1(k)\}\\ + \psi_4(k)k\{(1-2\sigma)I_0(k) + kI_1(k)\}]\sin kz\,dk,\\ [\tau_{rz}]_1 = -\frac{2(2-\sigma)}{(1+z^2)^{3/2}} + \frac{3}{(1+z^2)^{5/2}} + \int_0^\infty k^3[\psi_3(k)I_1(k)\\ + \psi_4(k)\{kI_0(k) + 2(1-\sigma)I_1(k)\}]\cos kz\,dk.\end{aligned} \tag{12}$$

These stresses are annulled throughout the surface of the cylinder provided that

$$\begin{aligned}\psi_3(k)\{kI_0(k) - I_1(k)\} + \psi_4(k)k\{(1-2\sigma)I_0(k) + kI_1(k)\}\\ = -\frac{2}{\pi k^2}\int_0^\infty \left\{\frac{(1-2\sigma)z}{(1+z^2)^{3/2}} - \frac{3z}{(1+z^2)^{5/2}}\right\}\sin kz\,dz,\\ \psi_3(k)I_1(k) + \psi_4(k)\{kI_0(k) + 2(1-\sigma)I_1(k)\}\\ = \frac{2}{\pi k^3}\int_0^\infty \left\{\frac{2(2-\sigma)}{(1+z^2)^{3/2}} - \frac{3}{(1+z^2)^{5/2}}\right\}\cos kz\,dz.\end{aligned} \tag{13}$$

Similarly, we find in terms of ψ_1 and ψ_2,

$$\begin{aligned}\psi_3(k) &= \{k^2 - 2(1-\sigma)(1-2\sigma)\}\psi_2(k)/k,\\ \psi_4(k) &= \{\psi_1(k) + 4(1-\sigma)\psi_2(k)\}/k.\end{aligned} \tag{14}$$

In order to render the integral convergent at the lower limit, the function χ_2 is to be modified as follows:

$$\begin{aligned}\chi_2 = a^3\rho + a^3\int_0^\infty [\{\psi_3(k)I_0(kr) + \psi_4(k)krI_1(kr)\}\cos kz\\ + f_2(k)r^2 + f_3(k)(1 - \tfrac{1}{2}k^2z^2)]\,dk,\end{aligned} \tag{15}$$

where

$$\begin{aligned} f_2(k) &= \frac{2}{\pi(1+\sigma)}\left\{\sigma K_0(k) - \frac{1-\sigma}{k}K_1(k)\right\}, \\ f_3(k) &= \frac{4(1-\sigma)(1-2\sigma)}{\pi(1+\sigma)k^2}\left\{K_0(k) + \frac{2}{k}K_1(k)\right\}. \end{aligned} \tag{16}$$

It can be shown that the derivative of the function χ_2 with respect to z gives no stress at z infinity.

It is equally obvious that odd derivatives of χ_2 with respect to z also gives functions with the desired properties. The set of functions is therefore

$$\frac{\partial}{\partial z}\chi_2\,, \frac{\partial^3}{\partial z^3}\chi_2\,, \frac{\partial^5}{\partial z^5}\chi_2\,, \cdots .$$

Since the use of any constant multiplier does not affect the desired properties, we may write the two sets of functions as follows:

$$\begin{aligned} \omega_0 &= -\chi_1\,, \qquad \omega_{2s} = -\frac{1}{(2s)!}\frac{\partial^{2s}}{\partial z^{2s}}\chi_1\,, \qquad (s \geq 1), \\ \omega'_{2s} &= -\frac{1}{(2s+1)!}\frac{\partial^{2s+1}}{\partial z^{2s+1}}\chi_2\,, \qquad (s \geq 0). \end{aligned} \tag{17}$$

In expressing the functions in terms of spherical coordinates, the following relations are useful[8].

$$\begin{aligned} I_0(kr)\sin kz &= \sum_{n=0}^{\infty}(-1)^n\frac{(k\rho)^{2n+1}}{(2n+1)!}P_{2n+1}(\mu), \\ kr\,I_1(kr)\sin kz &= \sum_{n=0}^{\infty}(-1)^n\frac{(k\rho)^{2n+1}}{(2n+1)!}\left\{\frac{2n(2n+1)}{4n+1} + \frac{(k\rho)^2}{4n+5}\right\}P_{2n+1}(\mu), \\ \frac{\partial^{2s}}{\partial z^{2s}}\log\frac{1+\mu}{1-\mu} &= -\frac{2(2s-1)!}{\rho^{2s}}P_{2s-1}(\mu), \\ \frac{\partial^{2s+1}\rho}{\partial z^{2s+1}} &= -\frac{(2s+1)!}{(4s+1)\rho^{2s}}\{P_{2s-1}(\mu) - P_{2s+1}(\mu)\} \end{aligned} \tag{18}$$

and in particular

$$\frac{\partial\rho}{\partial z} = P_1(\mu),$$

where P_n are Legendre functions of the first kind of order n. Consequently, we have

$$\begin{aligned} \omega_0 &= -\frac{a^3}{2}\log\frac{1+\mu}{1-\mu} + a^3\sum_{n=0}^{\infty}({}^{2n}\alpha_0 + {}^{2n}\gamma_0\rho^2)\rho^{2n+1}P_{2n+1}(\mu), \\ \omega_{2s} &= \frac{a^3P_{2s-1}(\mu)}{2s\rho^{2s}} + a^3\sum_{n=0}^{\infty}({}^{2n}\alpha_{2s} + {}^{2n}\gamma_{2s}\rho^2)\rho^{2n+1}P_{2n+1}(\mu) \end{aligned} \tag{19}$$

---

[8]For the first relation, cf. Ex. 63, p. 362, T. M. MacRobert, *Spherical harmonics*, 2nd edition, 1947, Methuen. For the last two relations, see Exs. 21-22, p. 332, E. T. Whittaker and G. N. Watson, *Modern analysis*, 1927, Cambridge University Press; or see formula 36, p. 105, E. W. Hobson, *Theory of spherical and ellipsoidal harmonics*, 1931, Cambridge University Press.

and

$$\begin{aligned} \omega_0' &= -a^3P_1(\mu) + a^3 \sum_{n=0}^{\infty} ({}^{2n}\beta_0 + {}^{2n}\delta_0\rho^2)\rho^{2n+1}P_{2n+1}(\mu), \\ \omega_{2s}' &= a^3 \frac{P_{2s-1}(\mu) - P_{2s+1}(\mu)}{(4s+1)\rho^{2s}} + a^3 \sum_{n=0}^{\infty} ({}^{2n}\beta_{2s} + {}^{2n}\delta_{2s}\rho^2)\rho^{2n+1}P_{2n+1}(\mu), \end{aligned} \tag{20}$$

where

$$\begin{aligned} {}^{2n}\alpha_{2s} &= -\frac{(-1)^{n+s}}{(2n)!(2s)!} \int_0^{\infty} \left\{\frac{\psi_1(k)}{2n+1} + \frac{2n\psi_2(k)}{4n+1}\right\} k^{2n+2s+1}\, dk, \\ {}^{2n}\beta_{2s} &= \frac{(-1)^{n+s}}{(2n)!(2s+1)!} \int_0^{\infty} \left\{\frac{\psi_3(k)}{2n+1} + \frac{2n\psi_4(k)}{4n+1}\right\} k^{2n+2s+2}\, dk, \\ {}^{2n}\gamma_{2s} &= -\frac{(-1)^{n+s}}{(4n+5)(2n+1)!(2s)!} \int_0^{\infty} \psi_2(k) k^{2n+2s+3}\, dk, \\ {}^{2n}\delta_{2s} &= \frac{(-1)^{n+s}}{(4n+5)(2n+1)!(2s+1)!} \int_0^{\infty} \psi_4(k) k^{2n+2s+4}\, dk \end{aligned} \tag{21}$$

and in particular

$$\begin{aligned} {}^0\alpha_0 &= -\int_0^{\infty} \{k\psi_1(k) + f_1(k)\}\, dk, \\ {}^0\beta_0 &= \int_0^{\infty} k^2\{\psi_3(k) + f_3(k)\}\, dk. \end{aligned}$$

The stress function. Now, construct the required stress function χ as follows:

$$\chi = \chi_0 + T \sum_{s=0}^{\infty} (A_{2s}\omega_{2s} + B_{2s}\omega_{2s}'), \tag{22}$$

where A_{2s} and B_{2s} are coefficients of superposition to be determined from the remaining boundary conditions on the surface of the cavity; the factor T being introduced to render the coefficients dimensionless. The function χ_0 has been given in (2).

It is evident that the stress function thus constructed gives a uniform axial tension of T per unit area at both ends of the cylinder and at the same time gives no traction on the surface of the cylinder. Its expression in spherical coordinates is readily found as follows:

$$\begin{aligned} \chi = \frac{Ta^3\rho^3}{30(1+\sigma)} \{3P_1(\mu) + 2(1-5\sigma)P_3(\mu)\} &- \frac{1}{2} Ta^3 A_0 \log \frac{1+\mu}{1-\mu} \\ &+ Ta^3 \sum_{n=1}^{\infty} \left(\frac{A_{2n}}{2n} + \frac{B_{2n}}{4n+1} - \frac{B_{2n-2}\rho^2}{4n-3}\right) \frac{P_{2n-1}(\mu)}{\rho^{2n}} \\ &+ Ta^3 \sum_{n=0}^{\infty} (C_{2n} + D_{2n}\rho^2)\rho^{2n+1}P_{2n+1}(\mu), \end{aligned} \tag{23}$$

where

$$\begin{aligned} C_{2n} &= \sum_{s=0}^{\infty} ({}^{2n}\alpha_{2s}A_{2s} + {}^{2n}\beta_{2s}B_{2s}), \\ D_{2n} &= \sum_{s=0}^{\infty} ({}^{2n}\gamma_{2s}A_{2s} + {}^{2n}\delta_{2s}B_{2s}). \end{aligned} \tag{24}$$

This function gives the following normal and tangential stresses[9]

$$
\begin{aligned}
\sigma_\rho &= \frac{1}{a^3}\left\{(2-\sigma)\mu\frac{\partial}{\partial\rho} + \frac{\sigma(1-\mu^2)}{\rho}\frac{\partial}{\partial\mu}\right\}\nabla^2\chi - \frac{1}{a^3}\frac{\partial^2}{\partial\rho^2}\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi \\
&= \frac{1}{3}T + \frac{2}{3}TP_2(\mu) + T\sum_{n=0}^{\infty}\left[\left\{\frac{(2n+1)(2n+2)}{\rho^2}A_{2n}\right.\right. \\
&\quad \left. + \frac{(2n+1)(2n+2)(2n-2+4\sigma)}{(4n+3)\rho^2}B_{2n} - \frac{4n(2n^2+3n-\sigma)}{4n-1}B_{2n-2}\right\}\frac{1}{\rho^{2n+1}} \\
&\quad - \left\{\frac{2n(2n-1)(2n+1)}{\rho^2}C_{2n} - \frac{4n(4n-2)(3n-4n\sigma+1-\sigma)}{(4n-1)\rho^2}D_{2n-2}\right. \\
&\quad \left.\left. + \frac{(4n+2)(4n+5)(2n^2-n-1-\sigma)}{4n+3}D_{2n}\right\}\rho^{2n}\right]P_{2n}(\mu),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\tau_{\rho\phi} &= \frac{(1-\mu^2)^{1/2}}{a^3}\left\{-(1-\sigma)\left(\frac{\partial}{\partial\rho} + \frac{\mu}{\rho}\frac{\partial}{\partial\mu}\right)\nabla^2\chi + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\mu}\right)\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi\right\} \\
&= -\frac{1}{3}T(1-\mu^2)^{1/2}P_2'(\mu) + T(1-\mu^2)^{1/2}\sum_{n=1}^{\infty}\left[\left\{\frac{2n+2}{\rho^2}A_{2n}\right.\right. \\
&\quad \left. + \frac{(2n+2)(2n-2+4\sigma)}{(4n+3)\rho^2}B_{2n} - \frac{2(2n^2-1+\sigma)}{4n-1}B_{2n-2}\right\}\frac{1}{\rho^{2n+1}} \\
&\quad + \left\{\frac{(2n-1)(2n+1)}{\rho^2}C_{2n} - \frac{(4n-2)(3n-4n\sigma+1-\sigma)}{(4n-1)\rho^2}D_{2n-2}\right. \\
&\quad \left.\left. + \frac{(4n+5)(4n^2+4n-1+2\sigma)}{4n+3}D_{2n}\right\}\rho^{2n}\right]P_{2n}'(\mu).
\end{aligned} \tag{26}
$$

Note that in spherical coordinates,

$$
\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial}{\partial\mu}\right\}. \tag{27}
$$

The remaining boundary conditions to be satisfied are that on the surface of the cavity the normal and tangential stresses vanish identically. By inserting $\rho = \lambda$ into (25) and (26) and equating separately the coefficient of each Legendre function or its derivative to zero, a system of linear equations is obtained. The system of equations may be replaced by the following system

$$
\begin{aligned}
A_{2n}' = -\frac{1}{3}\delta_{0,n} &+ \frac{4}{7-5\sigma}\delta_{1,n} + 2n(n+1)(4n-1)C_{2n}' \\
&+ 2(4n+1)(4n^4+4n^3-n^2-n+1-\sigma^2)D_{2n}', \quad (n \geq 0),
\end{aligned} \tag{28}
$$

$$
B_{2n-2}' = \frac{5}{3(7-5\sigma)}\delta_{1,n} + (4n+1)C_{2n}' + (2n-1)(2n+1)(4n+3)D_{2n}', \quad (n \geq 1),
$$

[9] Cf. footnote [4] for the formulas of stresses in spherical coordinates.

where $\delta_{mn} = 1$ or 0, according as $m = n$ or $m \neq n$, and

$$
\begin{aligned}
A'_{2n} &= \frac{2n+2}{\lambda^{2n+3}} A_{2n} + \frac{(2n+2)(2n-2+4\sigma)}{(4n+3)\lambda^{2n+3}} B_{2n} , \\
B'_{2n} &= \frac{2}{(4n+3)\lambda^{2n+3}} B_{2n} , \\
C'_{2n} &= -\frac{(2n-1)(2n+1)\lambda^{2n-2}}{4n^2+2n-4n\sigma+1-\sigma} C_{2n} \\
&\qquad + \frac{4(2n-1)(3n-4n\sigma+1-\sigma)\lambda^{2n-2}}{(4n-1)(4n^2+2n-4n\sigma+1-\sigma)} D_{2n-2} , \\
D'_{2n} &= -\frac{(4n+5)\lambda^{2n}}{(4n+3)(4n^2+2n-4n\sigma+1-\sigma)} D_{2n} .
\end{aligned}
\tag{29}
$$

A formal solution by successive approximations is as follows: Write, for $n \geq 0$,

$$
A'_{2n} = \sum_{p=0}^{\infty} A'^{(p)}_{2n}, \qquad B'_{2n} = \sum_{p=0}^{\infty} B'^{(p)}_{2n}, \tag{30}
$$

where

$$
\begin{aligned}
A'^{(0)}_{2n} &= -\frac{1}{3}\delta_{0,n} + \frac{4}{7-5\sigma}\delta_{1,n} , \\
B'^{(0)}_{2n} &= \frac{5}{3(7-5\sigma)}\delta_{0,n}
\end{aligned}
\tag{31}
$$

and by iteration,

$$
\begin{aligned}
A'^{(p)}_{2n} &= 2n(n+1)(4n-1)C'^{(p-1)}_{2n} \\
&\qquad + 2(4n+1)(4n^4+4n^3-n^2-n+1-\sigma^2)D'^{(p-1)}_{2n}, \qquad (n \geq 0), \\
B'^{(p)}_{2n-2} &= (4n+1)C'^{(p-1)}_{2n} + (2n-1)(2n+1)(4n+3)D'^{(p-1)}_{2n},
\end{aligned}
\tag{32}
$$

in which $C'^{(p-1)}_{2n}$ and $D'^{(p-1)}_{2n}$ are computed from $A'^{(p-1)}_{2n}$ and $B'^{(p-1)}_{2n}$ by means of (29) and (24).

Naturally, the foregoing method of solving the linear equations is valid as long as the two series in (30) are both convergent. From physical consideration alone, it seems likely that there will be convergence if λ is less than unity. An analytic proof of convergence may be established by means of the method used by Howland[10] and Knight.[11] However, for the sake of brevity no further detail will be given here. The solution will be illustrated by numerical examples which follow.

Numerical examples. Numerical examples will be given for the cases $\lambda = \frac{1}{2}$ and $\frac{1}{4}$. The coefficients in (21) are first computed and shown in Table 1, in which the integrals are integrated numerically from Gregory's formula[12] with the aid of tables of Bessel functions. Here the Poisson's ratio σ is taken as 0.3. Note that ${}^0\alpha_{2s}$ and ${}^0\beta_{2s}$ are not

[10] R. C. J. Howland, *Stresses in a plate containing an infinite row of holes*, Proc. Roy. Soc. A**148**, 471-491, 1935.

[11] R. C. Knight, *On the stresses in a perforated strip*, Quart. J. Math., Oxford, **5**, 255-268, 1934.

[12] E. T. Whittaker and G. Robinson, *Calculus of observations*, 4th ed., 1948, Blackie, p. 143.

TABLE 1

Coeff.	$2s$	$2n = 0$	$2n = 2$	$2n = 4$	$2n = 6$	$2n = 8$	$2n = 10$
	0	—	−0.50722	0.058547	−0.0099477	0.0018554	−0.00036549
	2	—	0.90856	−0.32478	0.10476	−0.031758	0.0092718
${}^{2n}\alpha_{2s}$	4	—	−0.73705	0.50010	−0.26061	0.11653	−0.047229
	6	—	0.42356	−0.46333	0.35548	−0.22024	0.11819
	8	—	−0.20102	0.32370	−0.34379	0.28190	−0.19336
	10	—	0.08479	−0.18884	0.26537	−0.27813	0.23762
	0	—	−0.42035	0.12257	−0.032782	0.0084192	−0.0021294
	2	—	0.54584	−0.31634	0.14157	−0.055231	0.019772
${}^{2n}\beta_{2s}$	4	—	−0.40438	0.38274	−0.25723	0.14117	−0.067883
	6	—	0.22351	−0.31586	0.29768	−0.21899	0.13621
	8	—	−0.10406	0.20626	−0.26027	0.24757	−0.19309
	10	—	0.04351	−0.11520	0.18784	−0.22390	0.21446
	0	−0.35018	0.070237	−0.017677	0.0044470	−0.0011071	0.00027445
	2	0.37928	−0.25533	0.12212	−0.049231	0.017970	−0.0061446
${}^{2n}\gamma_{2s}$	4	−0.22980	0.29397	−0.22533	0.13319	−0.067055	0.030298
	6	0.10584	−0.21698	0.24384	−0.19880	0.13226	−0.076390
	8	−0.04185	0.12579	−0.19497	0.21005	−0.17863	0.12821
	10	0.01509	−0.06258	0.12819	−0.17653	0.18655	−0.16314
	0	−0.048364	0.089523	−0.039753	0.014241	−0.0046136	0.0014109
	2	0.16114	−0.19141	0.13036	−0.068389	0.030793	−0.012550
${}^{2n}\delta_{2s}$	4	−0.10336	0.18829	−0.18781	0.13694	−0.082175	0.043140
	6	0.04842	−0.12918	0.17907	−0.17402	0.13451	−0.088482
	8	−0.01938	0.07185	−0.13274	0.16617	−0.16093	0.12978
	10	0.00705	−0.03486	0.08297	−0.13012	0.15450	−0.14990

TABLE 2

λ	$2n$	A'_{2n}	B'_{2n}	C'_{2n}	D'_{2n}
	0	-3.3487×10^{-1}	3.5515×10^{-1}	—	-8.456×10^{-4}
	2	8.4785×10^{-1}	-6.037×10^{-3}	1.144×10^{-2}	-2.411×10^{-4}
$\frac{1}{2}$	4	-5.563×10^{-2}	5.318×10^{-4}	-8.070×10^{-4}	7.428×10^{-6}
	6	1.072×10^{-2}	-3.931×10^{-5}	5.261×10^{-5}	-2.897×10^{-7}
	8	-1.380×10^{-3}	2.669×10^{-6}	-3.204×10^{-6}	1.267×10^{-8}
	10	1.442×10^{-4}	......	1.913×10^{-7}	-5.920×10^{-10}
	0	-3.3312×10^{-1}	3.0966×10^{-1}	—	$+1.158 \times 10^{-4}$
	2	7.4303×10^{-1}	-2.283×10^{-4}	1.360×10^{-3}	-8.133×10^{-6}
$\frac{1}{4}$	4	-2.125×10^{-3}	5.645×10^{-6}	-2.655×10^{-5}	6.424×10^{-8}
	6	1.145×10^{-4}	-1.172×10^{-7}	4.604×10^{-7}	-6.475×10^{-10}
	8	-4.135×10^{-6}	2.240×10^{-9}	-7.412×10^{-9}	7.329×10^{-12}
	10	1.223×10^{-7}	......	1.163×10^{-10}	-8.861×10^{-14}

needed in computation for they are the coefficients of the trivial term $\rho P_1(\mu)$ or z. The coefficients A'_{2n}, B'_{2n}, C'_{2n} and D'_{2n} are then computed by the method of successive approximations for $\lambda = \frac{1}{2}$ and $\frac{1}{4}$ respectively. The results are shown in Table 2. In the case $\lambda = \frac{1}{2}$ the computation has been carried out up to the fifth approximation. The convergence appears to be more rapid when λ is smaller. The coefficients are readily converted to A_{2n}, B_{2n}, C_{2n} and D_{2n} by means of (29).

It is now rather straightforward to compute the stress at any point in the cylinder. The most important one, however, is the maximum stress occurring at the surface of the cavity across the minimum section where $(\rho, \mu) = (\lambda, 0)$. This stress is given by

$$\begin{aligned} \max \sigma_\phi = \frac{1}{a^3}\Bigg[(1-\sigma)\left(\frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\rho}\right)\nabla^2\chi + \frac{\partial^2}{\partial\rho^2}\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi \\ + \left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{\mu}{\rho^2}\frac{\partial}{\partial\mu}\right)\chi\Bigg]_{\lambda,0} \\ = T - T\sum_{n=0}^{\infty}\Bigg\{\frac{(2n+1)A_{2n} + (2n-1+2\sigma)B_{2n}}{\lambda^{2n+3}} \\ - 4n^2\lambda^{2n-2}C_{2n} - 2(2n^2+8n-4n\sigma+7-5\sigma)\lambda^{2n}D_{2n}\Bigg\}P'_{2n+1}(0), \end{aligned} \tag{33}$$

where

$$P'_{2n+1}(0) = (-1)^n\binom{2n}{n}\frac{2n+1}{2^{2n}}. \tag{34}$$

The following values are obtained.

λ	0,	$\frac{1}{4}$,	$\frac{1}{2}$
T^{-1} (max σ_ϕ)	2.045,	2.081,	2.359

The value in the limiting case $\lambda = 0$ is the known result of a large tension member having a spherical cavity, that is $(27-15\sigma)/(14-10\sigma)$.

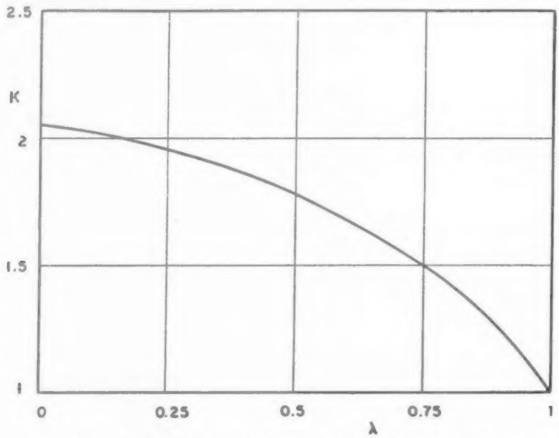

FIG. 2. Stress concentration factor K versus λ.

If we define a stress concentration factor K as the ratio of the maximum stress across the minimum section of the cylinder to the mean stress across the same section, then

$$K = (1 - \lambda^2)T^{-1} (\max \sigma_\phi). \tag{35}$$

Consequently, we have

λ	0,	$\frac{1}{4}$,	$\frac{1}{2}$,	1
K	2.045,	1.951,	1.769,	1

Here the value in the limiting case $\lambda = 1$ can be visualized readily from physical consideration of the cylinder. Figure 2 shows the results graphically.

The writer wishes to thank Mr. T. C. Lee for his invaluable assistance in preparing the manuscript.

BOOK REVIEWS

(*Continued from p. 370*)

Transactions of the symposium on computing, mechanics, statistics and partial differential equations. Vol. II. Symposium on Applied Mathematics Sponsored by The American Mathematical Society and Office of Ordnance Research, U.S. Army. 216 pp. $5.00.

These transactions of the second symposium on applied mathematics are reprinted from the Communications on Pure and Applied Mathematics Vol. VIII, No. 1 (1955). The eleven papers presented cover a wide range of topics, many in relatively new fields of interest or fields that lie in between well established ones. One very interesting feature of the group of papers is the surprising amount of overlap in the topics despite the tremendous scope of the conference. Indeed many of the papers are primarily descriptive surveys and seem to concentrate attention on the amalgamation of diverse topics into new fields.

The following lists the authors and the general topic or condensed title of the paper: P. M. Morse (operations research), J. Neyman (inductive inference), H. O. Hartley (analysis of variance), J. E. Mayer (statistical mechanics), M. R. Hestenes (iterative computation methods), J. Todd (numerical analysis), A. A. Bennett (ordnance computations), C. A. Truesdell (rate theory of elasticity), J. J. Stoker (stability of mechanical systems), F. Bureau (divergent integrals), and W. Feller (differential operators).

G. F. Newell

Engineering analysis. By D. W. Ver Planck and B. R. Teare, Jr. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1954. xii + 344 pp. $6.00.

This book is intended as an aid for students taking a course in "engineering analysis" such as the authors have been teaching for many years. The authors recognize that, throughout most engineering curricula, instructors try to present their students with carefully constructed and clearly worded problems whose solution will serve as exercises in acquiring familiarity with new principles and methods. In professional practice the engineer is never confronted with nicely stated problems, but rather with situations involving numerous uncertainties (physical, financial, and human) out of which he must construct his own problems and find their solution in a form which enables him to choose a definite plan of action. The sort of course which the authors teach is aimed at introducing the student to these facts of engineering life by posing him a series of practical situations, stated informally and even somewhat vaguely. The student must formulate precise problems, apply the basic principles of mechanics, thermodynamics, and electricity and magnetism, set up definite mathematical problems, and finally solve them in a usable way.

This book provides many interesting examples of such situations. It provides also many hints on how to attack them in an efficient and orderly way, as well as chapters reviewing the essential tools. One chapter summarizes basic physical principles (of mechanics, thermodynamics and heat flow, electricity and magnetism, and electric circuits). Another chapter reviews some basic mathematics, particularly standard elementary methods of solving ordinary differential equations. A final chapter gives useful suggestions on the interpretation and checking of mathematical results. The most interesting parts of the book will probably be those in which examples are studied in detail of the breaking down of complex situations into solvable mathematical problems whose answers enable the engineer to arrive at a decision. This is the most difficult part of professional practice, and one wonders how much can be learned by formal study and how much can be learned only from personal experience, more or less painful. However, there seems no doubt that the authors have made a valiant and commendable effort to help the student realize what sort of dilemmas will confront him in practice, and at the same time help him to develop confidence that with common sense, good working habits, and knowledge of some basic physical laws, he can make satisfactory progress in analyzing them.

P. S. Symonds

(*Continued on p. 464*)

TWO DIMENSIONAL SINK FLOW OF A VISCOUS, HEAT-CONDUCTING, COMPRESSIBLE FLUID; CYLINDRICAL SHOCK WAVES*

BY

T. YAO-TSU WU

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Introduction. The problem of the steady cylindrical source-type or sink-type flow has been of interest to fluid-dynamicists for several reasons. First, it is known that the corresponding problem of an inviscid compressible fluid has an exact solution containing a limit line of rather special type, namely, the sonic circle [2]. To the exterior of this circle the solution has two branches of values, one has its stagnation point at infinity (subsonic branch) and the other starts with maximum velocity at infinity (supersonic branch). Both of these two branches terminate at the limit line with infinite velocity gradient. Therefore the viscous and heat-conductive effects are expected to play an important role in continuing the solution further inward. Second, because of its cylindrical symmetry, this problem is one of the few nonlinear flows in more than one dimension for which there is only one independent variable, the radial distance. Consequently, the equations are simple enough to allow a unified discussion of the various effects. These are perhaps the reasons why this problem has attracted the attention of several authors [3, 4, 5].

In the first part of this investigation the Navier-Stokes equations are given for the cylindrical sink flow of a viscous, heat-conducting, perfect gas. Then, with some simplifying assumptions, the qualitative properties of the solutions are discussed in detail for the case of large Reynolds number Re. The definition of Re is $Re = \rho_1 a_1 r_1/\mu_1$, where $r = r_1$ locates the inviscid sonic circle with sonic speed a_1, and ρ_1, μ_1 are the fluid density and viscosity coefficient at $r = r_1$. These basic properties of the solutions thus comprehended serve as a useful guide in our final calculation of the solution.

In the second part of this paper the detailed calculation of the solution is carried out for the case of large Re. It is found that there is no single expression available for the solution uniformly valid in the entire flow region. The calculation is then performed in three different regions characterized by the length r_1 and the parameter Re. For $r > r_1 + O(Re^{-2/3})$ the approximate solution is obtained by using the PLK method**. The result fails to be a good approximation for r too close to r_1. For $|r - r_1| < O(Re^{-2/3})$, a different similarity rule for the variables leads to a system of cylindrical transonic equations which governs the flow across the sonic region. These equations can be integrated analytically for each of the different order terms. The result shows that the solutions belonging to the supersonic branch all contain cylindrical shock-type flow in this transonic region. In other words, these solutions gradually deviate away from the inviscid supersonic branch, reach a minimum near $r = r_1$ and then approach asymptotically to the viscous subsonic branch. It is also found that the shock strength is of $O(Re^{-1/3})$ while the shock-thickness is of $O(Re^{-2/3})$, results which are quite different

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**This terminology was introduced by Prof. H. S. Tsien in a series of seminars, held in California Institute of Technology in 1954, in which the method due to Poincaré-Lighthill-Kuo was discussed.

from those of the plane normal shock. Within this region, the thermodynamic variables satisfy the isentropic relation up to the order $O(Re^{-1/3})$ but deviate from it by a quantity of $O(Re^{-2/3})$. The approximate solution for $r < r_1 - O(Re^{-2/3})$ is subsequently carried out. Finally, the entropy variation of the fluid, and the effect due to variation in viscosity coefficients, are discussed.

The corresponding source flow problem was previously solved, using a numerical method, by Sakurai [4]; a qualitative investigation on this problem was later elaborated on many points by Levey [5], by making use of some conventional methods in nonlinear differential equations. In the latter work, the orders of magnitude of many flow quantities of interest were estimated. The present investigation on the sink flow is not merely a special case of the cylindrical flow other than the source type, but also presents an improved method which is more powerful than those used in the previous works (e.g. Ref. [3, 4, 5]). The PLK-method applied to the outer region yields a set of reliable boundary values for the transonic region and thus enables all flow quantities of interest to be calculated quantitatively in all regions.

The author thanks Prof. H. S. Tsien for suggesting the problem and Profs. M. S. Plesset and C. R. DePrima for their assistance on many points.

1. The fundamental equations. Here we are concerned with the two-dimensional sink flow of a viscous, compressible, heat-conducting fluid with polar symmetry. The only independent variable is the radial distance r from the origin. The radial velocity, u, is the only velocity component and is always negative for sink flow. Let p, ρ, T, μ, μ', λ, R, C_v, C_p denote respectively the pressure, density, absolute temperature, coefficients of shear and bulk viscosity, heat conductivity, gas constant, specific heats at constant volume and pressure. The momentum equation is

$$\rho u \frac{du}{dr} = -\frac{dp}{dr} + \frac{d}{dr}\left[2\mu \frac{du}{dr} + \frac{2}{3}(\mu' - \mu)\frac{1}{r}\frac{d}{dr}(ru)\right] + 2\mu \frac{d}{dr}\left(\frac{u}{r}\right) \tag{1.1}$$

and the energy equation is

$$\rho u r \frac{d}{dr}\left(\frac{u^2}{2} + C_p T\right) = \frac{d}{dr}\left\{r\left[\lambda \frac{dT}{dr} + \mu \frac{du^2}{dr} + \frac{2}{3}(\mu' - \mu)\left(\frac{1}{2}\frac{du^2}{dr} + \frac{u^2}{r}\right)\right]\right\}. \tag{1.2}$$

The continuity equation can be written in the following form if m denotes the sink strength,

$$2\pi \rho u r = -m. \tag{1.3}$$

The equation of state is assumed to be that of a perfect gas,

$$p = R\rho T. \tag{1.4}$$

Equations (1.1 − 1.4) are a system of nonlinear differential equations for four variables u, p, ρ and T if μ, μ', λ and C_p are known functions of T.

To reduce the equations to nondimensional form, the following nondimensional quantities are introduced:

$$\begin{aligned} r^* &= r/r_1, \quad \eta = \log r^*, \quad w = -u/a_1, \quad \theta = T/T_1 = (a/a_1)^2, \\ p^* &= p/p_1, \quad \rho^* = \rho/\rho_1, \quad \mu^* = \mu/\mu_1, \quad \mu'^* = \mu'/\mu_1, \end{aligned} \tag{1.5}$$

where quantities with the subscript 1 are fictitious quantities which would occur at the local Mach number unity for nonviscous and non-heat-conducting gas. Thus, with γ equal to the ratio of specific heats, assumed constant throughout, the sonic speed a_1 at $r = r_1$ is given by

$$a_1^2 = \gamma p_1/\rho_1 \qquad \text{and} \qquad 2\pi\rho_1 a_1 r_1 = m. \tag{1.6}$$

The continuity equation then becomes

$$\rho^* w r^* = 1. \tag{1.7}$$

Here w is always positive for sink flow. The equation of state is now

$$p^* = \theta\rho^*. \tag{1.8}$$

Eliminating p and ρ in Eq. (1.1) by using (1.3) and (1.4), and introducing the non-dimensional quantities, with $\eta = \log r^*$ as the independent variable, we obtain

$$\frac{dw}{d\eta} + \frac{1}{\gamma}\left[\frac{d}{d\eta}\left(\frac{\theta}{w}\right) - \frac{\theta}{w}\right]$$
$$= -2\alpha^*\left\{\mu^*(1 + k)\left(\frac{d^2w}{d\eta^2} - w\right) + \left[(1 + k)\frac{dw}{d\eta} + kw\right]\frac{d\mu^*}{d\eta}\right\}, \tag{1.9}$$

where

$$\alpha^* \equiv (Re)^{-1} = \frac{\mu_1}{\rho_1 a_1 r_1} = \frac{2\pi\mu_1}{m}, \qquad \mu' = (1 + 3k)\mu. \tag{1.10}$$

α^* thus denotes the inverse of the Reynolds number and will be considered much smaller than unity throughout this paper; whereas k expresses the relation between the two viscosity coefficients. Stokes' assumption on the value of μ' states that

$$\mu' = 0 \qquad \text{or} \qquad k = -1/3. \tag{1.11}$$

As this assumption does not agree with observations for many kinds of fluid [6], condition (1.11) will not be imposed on the final calculation of the flow field.

By using again Eq. (1.3), the energy equation (1.2) can be integrated once to yield the following nondimensional form

$$\frac{w^2}{2} + \frac{\theta}{\gamma - 1} + \alpha^*\mu^*\left[\frac{\sigma^{-1}}{\gamma - 1}\frac{d\theta}{d\eta} + (1 + k)\frac{dw^2}{d\eta} + 2kw^2\right] = \frac{\gamma + 1}{2(\gamma - 1)}, \tag{1.12}$$

where σ denotes the Prandtl number

$$\sigma = C_p\mu/\lambda. \tag{1.13}$$

The integration constant on the right hand side of Eq. (1.12) is chosen as shown above so that the limit solution for vanishing viscosity agrees at large r with that of a nonviscous iso-energetic flow.

Equations (1.9) and (1.12) are the two equations for two unknowns w and θ. The boundary conditions for them can be determined by requiring that they tend to their respective inviscid solutions as $\eta \to \infty$ so that these two solutions can appropriately be compared later.

1.1 *The inviscid solution.* The solution for sink flow of a compressible inviscid gas can be literally obtained by putting $\alpha^* = 0$ in the above equations without justifying

the validity of such a simplification. The solution of this reduced system of equations is known to be

$$\frac{r}{r_1} = e^{\eta} = w^{-1}\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}w^2\right)^{-1/(\gamma-1)}, \tag{1.14}$$

and

$$\theta = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}w^2, \qquad p^* = (\rho^*)^{\gamma} = \theta^{\gamma/(\gamma-1)}. \tag{1.15}$$

The value of r_1 can be expressed in terms of stagnation state as

$$r_1 = \frac{m}{2\pi a_1\rho_1} = \frac{m}{2\pi a_0\rho_0}\left(\frac{\gamma+1}{2}\right)^{(1/2)(\gamma+1)/(\gamma-1)}, \tag{1.16}$$

where $a_0^2 = \gamma p_0/\rho_0$ and p_0, ρ_0 are the isentropic stagnation pressure and density. Equation (1.15) simply states the iso-energetic and isentropic relations.

This inviscid solution $w(r)$ given by Eq. (1.14) is plotted in Fig. 1. It gives no solution for $r < r_1$; but for $r > r_1$, w is a double-valued function of r. On one branch w tends to

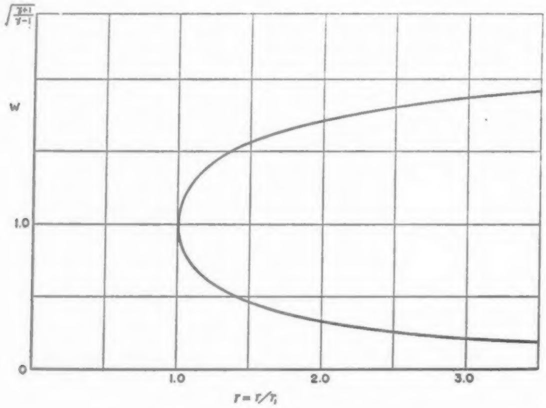

FIG. 1. Graph of inviscid solution.

zero so that thermodynamic variables tend to their stagnation values as $r \to \infty$; on the other branch w tends to the maximum speed attainable $[(\gamma+1)/(\gamma-1)]^{1/2}$, and the thermodynamic variables tend to zero as $r \to \infty$. They will be designated as subsonic and supersonic branch respectively. Both of these branches terminate at $r = r_1$ with sonic speed (at which the fluid speed equals the local speed of sound). The slope of the curve $w(r)$,

$$\frac{dw}{dr^*} = \frac{2}{\gamma+1}w^2(w^2-1)^{-1}\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}w^2\right)^{\gamma/(\gamma-1)},$$

is much smaller than unity for $r \gg r_1$ on both branches and consequently viscous effects become comparatively unimportant there. But as $r \to r_1$, $w \to 1$, dw/dr becomes numerically unbounded, and thus the viscous force near $r = r_1$ should play a role as significant as those of inertia and pressure forces.

Now this inviscid solution will be used as a guide to study the sink flow of a real fluid governed by Eqs. (1.9) and (1.12) for large values of Re, in the sense that it is assumed that the limit of the viscous solutions for vanishing viscosity approaches the inviscid solution as $r \to \infty$, for both subsonic and supersonic branches. By continuing these viscous solutions backward in r where viscous effects become more and more prominent, it is expected that the real fluid, affected by viscosity and heat conduction, will flow across this fictitious sonic circle, which is a limit line when viscosity is neglected.

It may be remarked here that the equations for source flow of real fluid can be obtained from (1.9) and (1.12) by changing the sign of the terms with factor α^* if w again represents the absolute value of the radial speed, normalized relative to a_1 [5]. Hence the inviscid solutions for source and sink flow are identical, but their respective viscous solutions will be shown later to have quite different features for all r.

2. Properties of the solution curves.

2.1. *Approximate differential equation in the "phase space".* In order to study the qualitative properties of the solution curves, several assumptions will be introduced in this section to simplify the analysis while most of the important features of the original system will still be maintained. The Prandtl number, σ, is assumed constant because μ and λ have almost the same dependence on temperature. In this section, μ is also taken to be constant so that $\mu^* = 1$. When the complete solution is calculated later, these assumptions introduced here become unnecessary.

Equation (1.12) can be integrated when the Prandtl number

$$\sigma = \tfrac{1}{2}(1 + 4\alpha^* k)/(1 + k) \tag{2.1}$$

(under Stokes' assumption, $k = -1/3$, then $\sigma = 3/4 - \alpha^*$), and the final integral is

$$\theta - \left[\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}(1 + 4\alpha^* k)w^2\right] = E \exp(-\sigma\eta/\alpha^*) = E\left(\frac{r}{r_1}\right)^{-\sigma/\alpha^*}, \tag{2.2}$$

where E is the integration constant. The value chosen above for σ is actually not far from experimental data ($\sigma \doteq 0.72$ for air at standard condition). As σ only appears in the coefficient of the derivative in Eq. (1.12), it follows from the theory of differential equations [Ref. 7, p. 142] that the solutions and all their derivatives will be continuous in σ for $w > 0$, $-\infty < \sigma < \infty$. Thus the assumption of choosing this particular value of σ would merely lead to simplification of analysis rather than any material change of the solutions. If we further require by physical argument that the deviation from the iso-energetic relation expressed by the term with the arbitrary constant E shall not overwhelm the left hand side terms for $r < r_1$, we may assume that $E = 0$. This restriction, however, can again be relaxed when the complete solution is discussed later. It will then be shown that E is indeed of the order $0(\alpha^*)$. Thus the particular solution with $E = 0$ would still provide a good approximation to the complete solution.

Substituting Eq. (2.2) with $E = 0$ into (1.9) and eliminating the explicit dependence on η by the substitution

$$V = -\frac{dw}{d\eta} = -r\frac{dw}{dr}, \tag{2.3}$$

we obtain

$$\alpha w^2 V \frac{dV}{dw} + V[1 - (1 - a\alpha)w^2] - w\{1 - [\beta + (a - 1)\alpha]w^2\} = 0, \tag{2.4}$$

where

$$\alpha = \frac{4\gamma}{\gamma + 1}(1 + k)\alpha^*, \qquad \beta = \frac{\gamma - 1}{\gamma + 1}, \qquad a = \frac{k}{1 + k}\,\frac{\gamma - 1}{\gamma}. \tag{2.5}$$

The variable V is closely related to the fluid velocity gradient. Since the term $a\alpha$ and $(a - 1)\alpha$ in the brackets are merely corrections to constant coefficients of $0(1)$, the properties of Eq. (2.4) would not be altered if we had neglected these terms in order to simplify further algebra. Thus the approximate differential equation

$$\alpha w^2 V \frac{dV}{dw} + V(1 - w^2) - w(1 - \beta w^2) = 0 \tag{2.6}$$

in the phase space (w, V) is expected to exhibit all important features of the original system, Eqs. (1.9 − 1.12), for $\mu^* = 1$. An equation similar to (2.6) was derived by Sakurai [4] and later was discussed in detail by Levey [5] for source flow in a real fluid.

2.2. *Properties of the solution curves in phase space.* Equation (2.6) is nonlinear and cannot be integrated. However, several important features of the solutions can be readily seen by studying the properties of the vector field (w, V) defined by Eq. (2.6), such as the type of its singular points, the curves of zero slope and zero curvature together with some obvious isoclines.

2.2a. *The curve of zero slope; the inviscid solution.* Let G_1 be the curve on which dV/dw given in Eq. (2.6) vanishes, G_1 is then given by

$$V = w(1 - \beta w^2)(1 - w^2)^{-1}, \tag{2.7}$$

which is also the inviscid solution in $w - V$ plane. The function $V_1(w)$ given by Eq. (2.7) has a simple pole at $w = 1$ (the fictitious sonic circle), and two zeros at $w = 0$ and $w = \beta^{-1/2}$ which correspond respectively to the subsonic and supersonic branch at $r = \infty$. Near the origin, $V_1(w)$ given by Eq. (2.7) has the following power series expansion

$$V_1 = w + (1 - \beta)w^3 + (1 - \beta)w^5 + (1 - \beta)w^7 + \cdots, \tag{2.8}$$

which starts from $w = 0$ with slope unity. Near the point $w = \beta^{-1/2}$, the expansion of $V_1(w)$ is

$$V_1 = \frac{2\beta x}{1 - \beta}\left[1 - \frac{1 + 3\beta}{2(1 - \beta)}(\beta^{1/2}x) + \frac{1 + 6\beta + \beta^2}{2(1 - \beta)^2}(\beta^{1/2}x)^2 - \frac{1 + 10\beta + 5\beta^2}{2(1 - \beta)^3}(\beta^{1/2}x)^3 + \cdots\right], \tag{2.9}$$

where

$$x = w - \beta^{-1/2}. \tag{2.9a}$$

This branch of the curve G_1 starts from $(\beta^{-1/2}, 0)$ with the slope $2\beta/(1 - \beta)$. The curve G_1 divides the infinite strip $0 \leq w \leq \beta^{-1/2}$ into regions of positive and negative slope as shown in Fig. 2. In this strip, dV/dw given in Eq. (2.6) becomes infinite only on (i) $V = 0$, $0 < w < \beta^{-1/2}$, and (ii) $w = 0$, $V \neq 0$. Besides, $w = \beta^{-1/2}$, $V \neq 0$ is also an isocline on which

$$dV/dw = (1 - \beta)/\alpha. \tag{2.10}$$

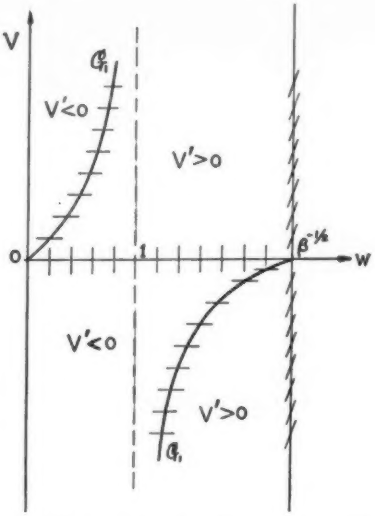

FIG. 2. Regions of positive and negative slopes; isoclines.

2.2b. *Properties of the singular points.* The only singular points of Eq. (2.6) are (0, 0) and $(\beta^{-1/2}, 0)$. The origin is a singular point of higher order. But if one sketches the vector field $(dw/dt, dV/dt)$ defined by

$$dw/dt = \alpha w^2 V, \qquad dV/dt = w(1 - \beta w^2) - V(1 - w^2)$$

along a simple closed curve in the neighborhood of the origin with the origin in its interior, one finds that the Poincaré index (see [8], p. 45) of this singularity is equal to -1. Thus the origin is a saddle point through which only two solution curves may pass. One of these is $w = 0$ which either yields a trivial solution $(V = 0)$ or has no physical meaning $(V \neq 0)$. The other solution curve starts from the origin with slope equal to 1 (which coincides there with the inviscid solution) and thus represents the only possible radial sink flow with stagnation at $r = \infty$. By substitution of a power series into Eq. (2.6) (or by ordinary iteration), the asymptotic value, for small α, of this solution near the origin is found to be

$$\begin{aligned} V \cong w &+ [(1 - \beta) - \alpha]w^3 + [(1 - \beta) - \alpha(5 - 4\beta) + 4\alpha^2]w^5 \\ &+ [(1 - \beta) - \alpha(14 - 16\beta + 3\beta^2) + 10\alpha^2(4 - 3\beta) - 27\alpha^3]w^7 + \cdots . \end{aligned} \tag{2.11}$$

The point $(\beta^{-1/2}, 0)$ is a singular point of regular type. In the neighborhood of $x \equiv w - \beta^{-1/2} = 0$, $V = 0$, Eq. (2.6) becomes

$$\frac{dV}{dx} = \frac{-2\beta x + (1 - \beta)V + P(x, V)}{\alpha V + Q(x, V)} \tag{2.12}$$

in which P and Q vanish like $x^2 + V^2$ as $x, V \to 0$. Thus the singularity [Ref. 8 pp. 37-44] is (i) a nodal point if $\alpha \leq (1 - \beta)^2/(8\beta)$, (for air, $\beta = 1/6$, $\alpha \leq 25/48$), or (ii) a spiral point if $\alpha > (1 - \beta)^2/(8\beta)$. As the problem will be confined to the case $\alpha \ll 1$,

we shall only consider this singularity to be a nodal point (which changes to a saddle point for source flow [5]). All solution curves passing through this point will have at this point two distinct slopes which can be calculated from the secular equation of Eq. (2.12), namely,

$$\begin{vmatrix} \lambda & -\alpha \\ 2\beta & \lambda - (1-\beta) \end{vmatrix} = 0, \tag{2.13}$$

which has two unequal positive roots

$$\lambda_1 = (1-\beta) - 2\alpha\beta/(1-\beta) + 0(\alpha^2), \qquad \lambda_2 = 2\alpha\beta/(1-\beta) + 0(\alpha^2). \tag{2.13a}$$

From these two eigen-values two eigen-vectors associated with Eq. (2.12) at $(x = 0, V = 0)$ can be obtained. By using this result it can be shown that the solution curves passing through $(x = 0, V = 0)$ have near this point the following parametric representation

$$V - \frac{2\beta}{1-\beta}x = C_1 e^{\lambda_1 t}, \qquad V - \left(\frac{1-\beta}{\alpha} - \frac{2\beta}{1-\beta}\right)x = C_2 e^{\lambda_2 t}. \tag{2.14}$$

Since $\lambda_1 > \lambda_2 > 0$, $V, x \to 0$ as $t \to -\infty$. It also follows from the above equations that there is an infinite number of solution curves, corresponding to arbitrary C_1 and C_2 $(C_2 \neq 0)$ which have the asymptotic value

$$V \cong \frac{2\beta}{1-\beta}x + C\,|\,x\,|^{(1-\beta)^2/(2\alpha\beta)} \qquad \text{near } (x = 0,\ V = 0); \tag{2.15a}$$

and, in addition, there is another solution curve $(\sim C_2 = 0)$ passing through this point, of the value

$$V \cong \left(\frac{1-\beta}{\alpha} - \frac{2\beta}{1-\beta}\right)x \qquad \text{near } (x = 0,\ V = 0). \tag{2.15b}$$

The first group of solutions, given by Eq. (2.15a), have the same limiting value at $r = \infty$ as the inviscid solution and hence represent the many possible sink flows starting with maximum velocity $w = \beta^{-1/2}$ at $r = \infty$, while the solution given by Eq. (2.15b) is physically irrelevant. The asymptotic value, for small α, of this physically significant solution near the point $x \equiv w - \beta^{-1/2} = 0$ is

$$\begin{aligned} V \cong \frac{2\beta x}{1-\beta}\Bigg\{&\left[1 + \frac{2\alpha\beta}{(1-\beta)^2} + \frac{8\alpha^2\beta^2}{(1-\beta)^4} + 0(\alpha^3)\right] \\ &- \frac{\beta^{1/2}x}{2(1-\beta)}\left[(1+3\beta) + 2\alpha\frac{\beta(3+13\beta)}{(1-\beta)^2} + 0(\alpha^2)\right] \\ &+ \frac{\beta x^2}{2(1-\beta)^2}\left[(1+6\beta+\beta^2) + 2(5+42\beta+33\beta^2)\frac{\alpha\beta}{(1-\beta)^2} + 0(\alpha^2)\right] + \cdots\Bigg\} \\ &+ C\,|\,x\,|^{(1-\beta)^2/2\alpha\beta}, \end{aligned} \tag{2.16}$$

where C is an arbitrary constant. The last term follows from Eq. (2.15a).

2.2c. *The curve of zero curvature.* The second derivative of the solution $V(w)$ can be deduced from (2.6). Let G_2 be the curve on which d^2V/dw^2 vanishes, the equation for G_2 is, except where wV is zero, given by:

$$2V^3 - w(1 + \beta w^2)V^2 + \alpha^{-1}(1 - \beta w^2)[V(1 - w^2) - w(1 - \beta w^2)] = 0. \quad (2.17)$$

The function $V_2(w)$ satisfying this equation has the following properties:

(i) For small values of α, V_2 has only one real value for either

$$0 \le w < 1 + \frac{3}{2}[(1 - \beta)\alpha/2]^{1/3} \qquad \text{or} \qquad w > \beta^{-1/2}\left[1 + \frac{\alpha}{4(1 - \beta)}\right];$$

V_2 has three real values for

$$1 + \frac{3}{2}[(1 - \beta)\alpha/2]^{1/3} \le w \le \beta^{-1/2}\left[1 + \frac{\alpha}{4(1 - \beta)}\right];$$

V_2 has two equal real values at

$$w = 1 + \frac{3}{2}[(1 - \beta)\alpha/2]^{1/3}; \qquad w = \beta^{-1/2}\left[1 + \frac{\alpha}{4(1 - \beta)}\right] \qquad \text{and} \qquad w = \beta^{-1/2}.$$

(ii) The curve G_2 starts from the origin with slope equal to unity and has, in the neighborhood of the origin, the following expansion:

$$\begin{aligned} V_2 \cong w &+ [(1 - \beta) - \alpha]w^3 + [(1 - \beta) - \alpha(5 - 4\beta) + 4\alpha^2]w^5 \\ &+ [(1 - \beta) - \alpha(14 - 16\beta + 3\beta^2) + 2\alpha^2(17 - 12\beta) - 21\alpha^3]w^7 + \cdots . \end{aligned} \quad (2.18)$$

(iii) The curve G_2 crosses $w = 1$ at

$$V_2(1) = (1 - \beta)^{2/3}(2\alpha)^{-1/3}[1 + 0(\alpha^{1/3})] \quad (2.19a)$$

with the slope

$$dV_2/dw = (2/3)(1 - \beta)^{1/3}(2\alpha)^{-2/3}[1 + 0(\alpha^{1/3})]. \quad (2.19b)$$

(iv) When $w = \beta^{-1/2}$, $V_2^2 = 0$, and $V_2 = \beta^{-1/2}$. Thus G_2 has a double point at $(\beta^{-1/2}, 0)$, where the curve has two different slopes $2\beta(1 - \beta)^{-1}$ $[1 + 0(\alpha)]$ and $(1 - \beta)\alpha^{-1}$ $[1 + 0(\alpha)]$. Near this point these two branches of the curve have the following expansions:

$$\begin{aligned} V_2^{(1)} \cong \frac{2\beta x}{(1 - \beta)}\Bigg\{&\left[1 + \frac{2\alpha\beta}{(1 - \beta)^2} + \frac{8\alpha^2\beta^2}{(1 - \beta)^4} + 0(\alpha^3)\right] \\ &- \frac{\beta^{1/2}x}{2(1 - \beta)}\left[(1 + 3\beta) + 4\alpha\frac{\beta(7\beta - 1)}{(1 - \beta)^2} + 0(\alpha^2)\right] \\ &+ \frac{\beta x^2}{2(1 - \beta)^2}[(1 + 6\beta + \beta^2) + 0(\alpha)] + \cdots\Bigg\} \end{aligned} \quad (2.20)$$

and

$$V_2^{(2)} \cong \frac{1 - \beta}{\alpha}x\left[1 - \frac{2\alpha\beta}{(1 - \beta)^2} + 0(\alpha^2)\right] + \cdots . \quad (2.21)$$

(v) The slope of the curve G_2 becomes infinite at

$$w \sim 1 + \tfrac{3}{2}[(1 - \beta)\alpha/2]^{1/3}, \qquad V \sim -\alpha^{-1/3}[(1 - \beta)/2]^{2/3}$$

and

$$w \sim \beta^{-1/2}\left[1 + \frac{\alpha}{4(1 - \beta)}\right], \qquad V \sim \frac{1}{2}\beta^{-1/2}.$$

(vi) For $w \gg 1$, $V_1 \sim \beta w$, $V_2 \sim \beta w^3/2$.

The curve G_2 divides the $(w - V)$ plane into regions of positive and negative curvatures as shown in Fig. 3.

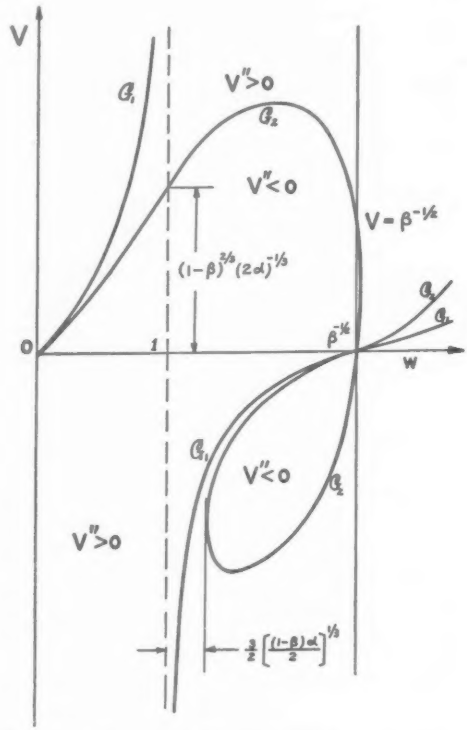

Fig. 3. Regions of positive and negative curvature.

2.2d. *Sketch of solution curves $V(w)$; definition of cylindrical shock.* The above discussions on isoclines, types of singularities, regions of positive and negative slope and curvature, enable the solution curves to be sketched.

First let us consider the solution curve starting from the origin. Comparison of Eqs. (2.8), (2.11) and (2.18) shows that, for α and w both small,

$$V_1(w) > V(w) > V_2(w), \tag{2.22}$$

the difference $(V_1 - V)$ being of $0(\alpha w^3)$ while $(V - V_2) \sim 0(\alpha^2 w^7)$. Thus the solution curve $V(w)$ lies in between G_1 and G_2 where the slope and curvature of $V(w)$ are both

positive. A careful study of the slope of $V(w)$ and G_2 indicates that the solution curve lies above G_2 for $w > 0$. Hence $V(w)$ is a monotonic increasing function of w with increasing slope, passing through $w = 1$ between the points $V = (1 - \beta)/\alpha$ and $V = (1 - \beta)^{2/3} (2\alpha)^{-1/3}$, and finally ending up at $w = \beta^{-1/2}$ with the slope $dV/dw = (1 - \beta)/\alpha$ (cf. Eq. 2.10 and Fig. 4).

As previously shown in Eq. (2.16), there is an infinite number of solution curves starting from ($w = \beta^{-1/2}$, $V = 0$) with the same slope $2\beta/(1 - \beta)$. However, for $(w - \beta^{-1/2})$ and α both small enough, comparison of Eqs. (2.9), (2.16) and (2.20) again shows that all these solutions satisfy, for small negative $(w - \beta^{-1/2})$, the following inequality

$$V(w) < V_2^{(1)}(w) < V_1(w), \tag{2.23}$$

as shown in Fig. 4. As w decreases from $\beta^{-1/2}$, V (for every finite C in Eq. 2.16) decreases with increasing slope until it intercepts G_2 with a positive slope. For further decrease

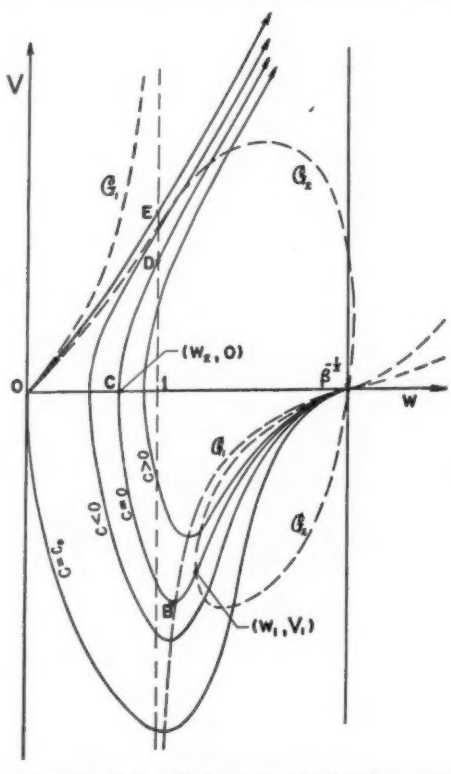

FIG. 4. Sketch of the solution curves in the phase space.

in w, the curve V should lie above a straight line with this positive slope at the point of intersection because V has positive curvature in this region. Hence the solution curves will eventually meet G_1 with zero slope. From there on, for further decrease in w, V increases from negative values and later crosses $V = 0$ with infinite slope at some point

in between $w = 0$ and $w = \beta^{-1/2}$, as can be shown by the method of bounding curves, and will be made explicit in our later calculation. Further extension of these solution curves shows that V increases with increasing w and finally approaches asymptotically to the subsonic branch solution which starts from the origin. There is a particular value of the integration constant C (in Eq. 2.16), say, $C = C_0 < 0$, for which the solution curve finally ends up at the origin with infinite slope. For $C < C_0$, the solution ceases to have physical meaning. On the other hand, the solution curves for $C > C_0$ have a very interesting feature that these viscous solutions all exhibit the transition process from the *inviscid* supersonic branch toward the *viscous* subsonic branch. Let us consider, in particular, the solution with $C = 0$. It first intercepts G_2 at (w_1, V_1), say, and then crosses $V = 0$ at $w = w_2$. Since $w_1 > 1$ and $w_2 < 1$ (as will be shown later), the flow between these two states may thus be defined as that of a "cylindrical shock".* Inspired by the result obtained in Eqs. (2.19a, b), we see that the equation governing such a cylindrical shock flow can be approximated by the following similarity transformation

$$w = 1 + \alpha^{1/3}\bar{w}, \qquad V = \alpha^{-1/3}\bar{v}, \tag{2.24}$$

which will render all terms in Eq. (2.6) equally important in this flow region, that is, for vanishing α,

$$\bar{v}\, d\bar{v}/d\bar{w} = 2\bar{w}\bar{v} + (1 - \beta). \tag{2.25}$$

Since this equation governs the flow of both branches near the sonic speed, it may be called the "equation for cylindrical transonic flow".

2.3. *Sketch of solution curves $w(\eta)$ in physical space.* From the definition of $V(w)$ given by Eq. (2.3), we have

$$\eta = -\int^{w} [V(w)]^{-1}\, dw + h, \tag{2.26}$$

where the integral stands for an indefinite integral and

$$h = \frac{1}{\gamma - 1} \log\left(\frac{2}{\gamma + 1}\right) \tag{2.26a}$$

so that η tends to its inviscid solution for η large.

Now for the subsonic branch, $V(w) \geq 0$, hence η is a monotonically decreasing function of w. Moreover, for some value of w, $V(w)$ is less than its corresponding inviscid solution, $V_1(w)$ (see Eq. 2.22). Hence, from Eq. (2.26),

$$\eta_{\text{vis}}(w) < \eta_{\text{invis}}(w). \tag{2.27}$$

In other words, at every η, $(w)_{\text{vis}}$ is slowed down from its inviscid value due to the viscous effect.

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*This terminology is adopted by both Sakurai [4] and Levey [5] to describe such type of flow. The term "shock" is borrowed from its conventional meaning to indicate the transition from one branch to the other, though the transition is rather different from that occurring in a plane normal shock. Perhaps this terminology relates more closely to the conventional meaning of a shock for the constant C slightly greater than C_0 (see Fig. 4), because then the jump in w and the slope $dw/d\eta$ in transition become greater, and the position of transition is farther out from $r = r_1$ (see Fig. 5). But since there is no adequate criterion to distinguish one from another value of C, we shall retain this name. Another terminology, the dissipation layer, is suggested by Prof. H. S. Tsien to avoid this ambiguity and, in addition, to stress the importance of viscous effects in this layer.

For the supersonic branch starting from $w = \beta^{-1/2}$, $V(w) \leq 0$ for $w_2 \leq w \leq \beta^{-1/2}$, hence in this interval η is a monotonically increasing function of w. At $w = w_2$, ($\sim \eta = \eta_2$, say), $(dw/d\eta) = -V(w_2) = 0$, and, from Eq. (2.6), $(d^2w/d\eta^2) = (VdV/dw)_{v=0} = (1 - \beta w_2)/(\alpha w_2) > 0$, therefore $w(\eta_2) = w_2$ is the only minimum of w on this branch. For $\eta \leq \eta_2$, η decreases with increasing w. Furthermore, because of the relation given in Eq. (2.23), we have

$$\eta_{\text{vis}}(w) > \eta_{\text{invis}}(w) \qquad \text{for} \qquad w > 1,\ V \leq 0. \tag{2.28}$$

The above properties of the solution enable the solution curves to be sketched, as shown in Fig. 5. As the constant C decreases from zero, the minimum value decreases and the jump in w increases while the jump takes place farther upstream.

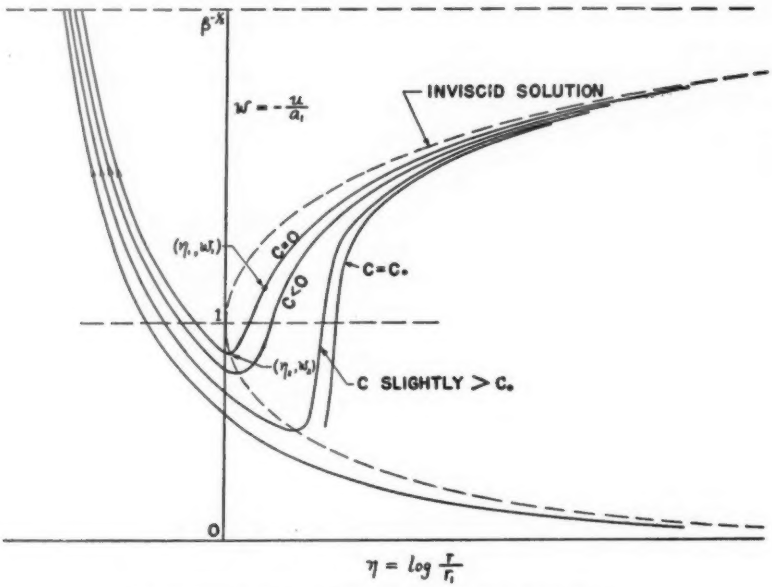

FIG. 5. Sketch of the solution curves in the physical space.

It follows from Eq. (2.24) that if we introduce

$$\eta = \alpha^{2/3}\xi, \qquad w = 1 + \alpha^{1/3}\bar{w}, \tag{2.29}$$

then Eq. (2.25) becomes

$$\frac{d^2\bar{w}}{d\xi^2} = -2\bar{w}\,\frac{d\bar{w}}{d\xi} + (1 - \beta), \tag{2.30}$$

which is the equation governing the flow near $w = 1$, $\eta = 0$. Integrating this equation once, we obtain

$$d\bar{w}/d\xi + \bar{w}^2 = (1 - \beta)\xi + \text{const.} \tag{2.31}$$

This equation will be integrated and discussed in our final calculation.

3. Calculation of the solutions by using *PLK*-method*. In this section we shall calculate $w(\eta)$, $\theta(\eta)$ governed by the original system of Eqs. (1.9-1.12). Throughout this section μ will again be assumed constant so that $\mu^* = 1$, but no restriction will be imposed on σ and k. Consequently Eqs. (1.9) and (1.12) become

$$w^2 \frac{dw}{d\eta} + \frac{1}{\gamma}\left[w \frac{d\theta}{d\eta} - \theta \frac{dw}{d\eta} - \theta w\right] = -\frac{\gamma + 1}{2\gamma}\alpha\left(\frac{d^2w}{d\eta^2} - w\right)w^2, \tag{3.1}$$

$$\theta + \frac{\gamma - 1}{2}(1 + b\alpha)w^2 + \frac{\gamma + 1}{4\gamma}\alpha\left[\frac{\sigma^{-1}}{1 + k}\frac{d\theta}{d\eta} + (\gamma - 1)\frac{dw^2}{d\eta}\right] = \frac{\gamma + 1}{2}, \tag{3.2}$$

where α is given by Eq. (2.5) and

$$b = a/\beta = (1 + 1/\gamma)(1 + 1/k)^{-1}. \tag{3.3}$$

Using the PLK-method, as described below, the generalization to the case $\mu = \mu(T)$ presents no particular difficulty. The calculation for the case when μ is proportional to local temperature is carried out in Ref. 1, Sec. 7. The result shows that all important features of flow quantities for $\mu^* = 1$ remain up to the term of $0(\alpha^{2/3})$.

It was seen before that, even for a simplified version of these equations, such as Eq. (2.6), the conventional perturbation method merely leads to asymptotic solutions for small α near $w = 0$ or $w = \beta^{-1/2}$ (see Eqs. 2.11, 2.16) because the coefficient of w^n does not diminish as $n \to \infty$. This asymptotic result fails to be a good approximation to the required solution as w deviates further from $w = 0$ or $w = \beta^{-1/2}$ and becomes almost useless for calculations near $w = 1$. Now let us resort to the PKL-method, which is in essence to expand the solution in terms of power series in α with coefficients as undetermined functions of a parameter ξ.

$$\begin{aligned} w(\xi) &= \xi + \alpha w^{(1)}(\xi) + \alpha^2 w^{(2)}(\xi) + \cdots, \\ \eta(\xi) &= \eta^{(0)}(\xi) + \alpha\eta^{(1)}(\xi) + \alpha^2\eta^{(2)}(\xi) + \cdots, \\ \theta(\xi) &= \theta^{(0)}(\xi) + \alpha\theta^{(1)}(\xi) + \alpha^2\theta^{(2)}(\xi) + \cdots. \end{aligned} \tag{3.4}$$

The need of a parameter ξ to represent the solution and that w starts with the term ξ are clearly suggested by our previous discussions. Substituting these expansions into Eqs. (3.1) and (3.2), noting that

$$dw/d\eta = w'/\eta', \qquad d\theta/d\eta = \theta'/\eta', \qquad d^2w/d\eta^2 = w''(\eta')^{-2} - w'\eta''(\eta')^{-3}, \tag{3.5}$$

where prime stands for $d/d\xi$, and then equating equal powers of α, we obtain the zeroth order equations as follows:

$$\left[\gamma\xi^2 + \xi^2 \frac{d}{d\xi}(\theta^{(0)}/\xi) - \xi\theta^{(0)}\eta^{(0)\prime}\right](\eta^{(0)\prime})^2 = 0, \tag{3.6a}$$

$$\left[\theta^{(0)} - \left(\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\xi^2\right)\right]\eta^{(0)\prime} = 0. \tag{3.6b}$$

If we choose $\eta^{(0)\prime}$ different from zero, then we have the zeroth order solution

$$\theta^{(0)}(\xi) = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\xi^2, \tag{3.7a}$$

$$\eta^{(0)}(\xi) = -\log\xi - \frac{1}{\gamma - 1}\log\left|\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\xi^2\right|. \tag{3.7b}$$

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*See the footnote in the Introduction.

The integration constant in Eq. (3.7b) has been so chosen that when $\alpha = 0$, $\xi \equiv w$ and $\eta^{(0)}(w)$ agrees with inviscid solution.

The coefficient of α in the expanded equations gives the following first order equations:

$$\theta^{(1)} + (\gamma - 1)\xi w^{(1)} = -\frac{\gamma - 1}{2} b\xi^2 + \frac{\gamma + 1}{2}\epsilon\xi^2(1 - \beta\xi^2)(1 - \xi^2)^{-1}, \tag{3.8a}$$

$$\begin{aligned}(\eta^{(0)\prime})^2\Big\{(2\gamma\xi + \theta^{(0)})w^{(1)} + (\gamma\xi^2 - \theta^{(0)})w^{(1)\prime} + \xi^2\frac{d}{d\xi}(\theta^{(1)}/\xi) - \xi\theta^{(0)}\eta^{(1)\prime} \\ - (\xi\theta^{(1)} + w^{(1)}\theta^{(0)})\eta^{(0)\prime}\Big\} = \frac{\gamma + 1}{2}\xi^2[\eta^{(0)\prime\prime} + \xi(\eta^{(0)\prime})^3],\end{aligned} \tag{3.8b}$$

where b is given by Eq. (3.3) and

$$\epsilon = \frac{\gamma - 1}{\gamma}\left[1 - \frac{1}{2\sigma(1 + k)}\right]. \tag{3.8c}$$

Substituting Eqs. (3.7) and (3.8a) into (3.8b), and rearranging the terms, we finally obtain [see Ref. 1, p. 25]:

$$\begin{aligned}&\frac{d}{d\xi}[\eta^{(0)\prime}w^{(1)} - \eta^{(1)}] \\ &\quad = (1 - \epsilon)\left[\frac{2\xi}{(1 - \xi^2)^2} + \frac{\gamma\xi}{\xi^2 - 1}\right] + A\frac{2\beta\xi}{(\beta\xi^2 - 1)^2} + B\frac{2\beta\xi}{(\beta\xi^2 - 1)},\end{aligned} \tag{3.9}$$

where

$$A = \frac{1}{2\beta^2}(1 - a)(1 - \beta), \qquad B = \frac{1}{2\beta}\left[\frac{1 - a}{\beta} - (a - \epsilon) - \frac{(1 + \epsilon)2\beta}{1 - \beta}\right] \tag{3.9a}$$

and the constants a, b, ϵ are given by Eqs. (3.3) and (3.8c). Now in order to ascribe to $w^{(1)}$ a possibly lowest order singularity at $\xi = 1$ to improve the convergence of the series, we decompose Eq. (3.9) as follows:

$$\frac{d}{d\xi}[\eta^{(0)\prime}w^{(1)}] = A\frac{2\beta\xi}{(\beta\xi^2 - 1)^2} + B\frac{2\beta\xi}{(\beta\xi^2 - 1)}, \tag{3.10a}$$

$$\frac{d}{d\xi}\eta^{(1)} = -(1 - \epsilon)[2\xi(\xi^2 - 1)^{-2} + \gamma\xi(\xi^2 - 1)^{-1}]. \tag{3.10b}$$

The solution of Eqs. (3.10a, b) is then

$$w^{(1)}(\xi) = -A\xi(1 - \xi^2)^{-1} - B\xi(1 - \beta\xi^2)(1 - \xi^2)^{-1}\log\left|\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\xi^2\right|, \tag{3.11a}$$

$$\eta^{(1)}(\xi) = -(1 - \epsilon)\left[(1 - \xi^2)^{-1} + \frac{\gamma}{2}\log|1 - \xi^2|\right]. \tag{3.11b}$$

The integration constant of Eq. (3.9) is first absorbed in $\eta^{(1)}$ and is then omitted because of its negligible contribution near $\xi = 1$. In the above first order solution, $w^{(1)}$ and $\eta^{(1)}$ have singularities at $\xi = 1$ of the same order. After $w^{(1)}$ is so determined, $\theta^{(1)}(\xi)$ is then given by Eq. (3.8a).

Proceeding in a similar manner to obtain the second order equations by equating terms with α^2, we find that the resulting equations possess solution of quite lengthy ex-

pression, in which $\eta^{(2)}(\xi)$ starts with the term $2(1-\beta)(1-\epsilon)(1-\xi^2)^{-4}$ followed by terms of $0((1-\xi^2)^{-3})$, while $w^{(2)}(\xi)$ still can be made to be of $0((1-\xi^2)^{-1})$. Therefore the final solution can be expressed parametrically as follows:

$$w(\xi) = \xi - \alpha\left[A\frac{\xi}{1-\xi^2} + B\frac{\xi(1-\beta\xi^2)}{1-\xi^2}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right] + 0\left(\frac{\alpha^2}{1-\xi^2}\right), \tag{3.12a}$$

$$\begin{aligned}\eta(\xi) = &-\left[\log\xi + \frac{1}{\gamma-1}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right]\\ &- \alpha(1-\epsilon)\left[\frac{1}{1-\xi^2} + \frac{\gamma}{2}\log|1-\xi^2|\right] + \alpha^2\frac{2(1-\beta)(1-\epsilon)}{(1-\xi^2)^4}\\ &+ 0\left(\frac{\alpha^2}{(1-\xi^2)^3}, \frac{\alpha^3}{(1-\xi^2)^7}\right) + C\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|^{(1-\beta)^2/(2\alpha\beta)},\end{aligned} \tag{3.12b}$$

where A, B are constants given in Eqs. (3.9a) and C is an arbitrary constant for the solution starting from $\xi = \beta^{-1/2}$, but $C = 0$ for the solution starting from $\xi = 0$. The term with C is included in the above equation in order to exhibit the behavior of solution curves near the nodal point ($w = \beta^{-1/2}$, $\eta = \infty$), as previously discussed in Sec. 2.2b. This term is of higher order than any finite power of α. In fact, it has an essential singularity at $\alpha = 0$ and hence cannot be obtained by the expansion procedure. The value of θ is given by

$$\begin{aligned}\theta(\xi) = &\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right)\\ &+ \alpha(\gamma-1)\left[A\frac{\xi^2}{1-\xi^2} + B\frac{\xi^2(1-\beta\xi^2)}{1-\xi^2}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right.\\ &\left.-\frac{b}{2}\xi^2 + \frac{\epsilon}{2\beta}\frac{\xi^2(1-\beta\xi^2)}{1-\xi^2}\right] + 0\left(\frac{\alpha^2}{(1-\xi^2)^2}\right).\end{aligned} \tag{3.12c}$$

Several interesting features of the above solution may be mentioned here. (i) $\epsilon = 0$ for $\sigma = [2(1+k)]^{-1}$ ($= 3/4$ if $k = -1/3$, see Eq. 3.8c). With the value of σ and k lying in the experimental range, ϵ is still a small number. Consequently, the variation in σ and k only contribute a small correction to the coefficients of $0(1)$ in the solution. This fact confirms our previous statement in Sec. 2.1. (ii) By substituting Eq. (3.12) into Eq. (2.2), it can be found that the arbitrary constant E is of $0(\alpha\epsilon)$. (iii) The most important property of the above solution is that it does not provide an approximate solution with uniform accuracy over the interval of ξ such that $0 \le w \le \beta^{-1/2}$. As ξ approaches unity, the higher order terms, especially in $\eta(\xi)$, become more important relative to the zeroth order term. More precisely, the solution is good only for $0 \le \xi \le 1 - K\alpha^{1/3}$ and $1 + K\alpha^{1/3} \le \xi \le \beta^{-1/2}$, K being a positive constant of $0(1)$. At $\xi = 1 \pm K\alpha^{1/3}$, all terms in the expression for $\eta(\xi)$ become of the same order, $0(\alpha^{2/3})$; but the convergence can be made sufficiently rapid by an appropriate choice of the value K.

In the subsequent calculation of the solution through the transonic region, we shall only consider a particular solution with $C = 0$ and $\epsilon = 0$ in Eqs. (3.12). Furthermore, it

has been found convenient to take $K = 2(\gamma + 1)^{-1/3}$. With this value of K, we obtain, from Eqs. (3.12), the following result:

(i) on supersonic branch, at $w = 1 + 2\alpha^{1/3} (\gamma + 1)^{-1/3}$,

$$\eta = 2.28(\gamma + 1)^{1/3}\alpha^{2/3} \quad \text{and} \quad dw/d\eta = 0.615\alpha^{-1/3}(\gamma + 1)^{-2/3}; \tag{3.13a}$$

(ii) on subsonic branch, at $w = 1 - 2\alpha^{1/3} (\gamma + 1)^{-1/3}$,

$$\eta = 1.766(\gamma + 1)^{1/3}\alpha^{2/3} \quad \text{and} \quad dw/d\eta = -0.477\alpha^{-1/3}(\gamma + 1)^{-2/3}. \tag{3.13b}$$

These values will serve for the boundary conditions imposed on the transonic solution to be obtained below.

That the PLK-method is powerful in solving this problem can still be stressed further by the following argument. As the first order term in the expression for $w(\xi)$ (see 3.12a) is quite unimportant in the aforementioned regions of ξ, one perhaps would try, instead of Eq. (3.4), a simpler expansion

$$\eta(w) = \eta^{(0)}(w) + \alpha\eta^{(1)}(w) + \cdots \tag{3.14}$$

and a similar expansion for θ in terms of w. It can be shown that the above expansion will yield a solution in which $\eta^{(0)}$ is identical to inviscid solution, but $\eta^{(1)}$ has, in addition to a simple pole at $w = 1$, a pole and a logarithmic singularity at $w = \beta^{-1/2}$. Consequently the assumed expansion (3.14) becomes invalid for r large on the supersonic branch, and thus leads to an erroneous result.

4. The solution in the transonic region. To obtain an approximate solution in the transonic region, as discussed in Secs. 2.2d, 2.3 (see Eqs. 2.24, 2.25 and 2.29) and also as guided by the boundary conditions (3.13), first we distort the independent variable by the transformation

$$\eta = \alpha^{2/3}\xi \tag{4.1}$$

and then expand w, θ into the form:

$$w(\xi) = 1 + \alpha^{1/3}w^{(1)}(\xi) + \alpha^{2/3}w^{(2)}(\xi) + \alpha w^{(3)}(\xi) + \cdots, \tag{4.2}$$

$$\theta(\xi) = 1 + \alpha^{1/3}\theta^{(1)}(\xi) + \alpha^{2/3}\theta^{(2)}(\xi) + \alpha\theta^{(3)}(\xi) + \cdots. \tag{4.3}$$

With $\epsilon = 0$ ($\sim 2\sigma(1 + k) = 1$), Eqs. (3.1) and (3.2) reduce to

$$[1 - (1 - a\alpha)w^2]\frac{dw}{d\eta} + w\{1 - [\beta + (a - 1)\alpha]w^2\} = \alpha w^2 \frac{d^2w}{d\eta^2}, \tag{4.4}$$

$$\theta = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}(1 + b\alpha)w^2 + 0(\alpha^2). \tag{4.5}$$

Substituting (4.1) and (4.2) into (4.4), we obtain the first order equation:

$$\frac{d^2w^{(1)}}{d\xi^2} + 2w^{(1)}\frac{dw^{(1)}}{d\xi} = (1 - \beta) \tag{4.6}$$

and the second order equation:

$$\frac{d^2w^{(2)}}{d\xi^2} + 2\frac{d}{d\xi}(w^{(1)}w^{(2)}) = \frac{d}{d\xi}(w^{(1)})^3 - (1 + \beta)w^{(1)}. \tag{4.7}$$

The correction due to the terms with constants a and b enters only in $w^{(3)}$ and higher order terms.

Now Eq. (4.6) can be integrated once to yield:

$$\frac{dy}{dx} + y^2 = x + x_1 , \tag{4.8}$$

where

$$y = \left(\frac{\gamma + 1}{2}\right)^{1/3} w^{(1)}, \qquad x = \left(\frac{\gamma + 1}{2}\right)^{-1/3} \xi, \tag{4.9}$$

and the integration constant x_1 can be determined by using the boundary condition (3.13) to give

$$\begin{aligned} x_1 &= 0.033 \text{ for the supersonic branch,} \\ x_1 &= -\ 0.0056 \text{ for the subsonic branch.} \end{aligned} \tag{4.10}$$

The nonlinear equation (4.8) (see also Eq. (2.31)) is an approximation to the Navier-Stokes equation, a case in which all the inertia, pressure and viscous forces are equally important. Now Eq. (4.8) is of Riccati type, which, by using the transformation

$$y(x) = \frac{1}{v}\frac{dv}{dx} , \tag{4.11}$$

can be reduced to a second order linear equation:

$$\frac{d^2 v}{dx^2} - (x + x_1)v = 0. \tag{4.12}$$

The solution of this equation for $(x + x_1) > 0$ is

$$v(z) = Mz^{1/3}[I_{-1/3}(z) + NI_{1/3}(z)], \qquad z = \tfrac{2}{3}(x + x_1)^{3/2}, \tag{4.13}$$

where $I(z)$ is the modified Bessel function of the first kind, and M, N are the integration constants. By using Eq. (4.11), the solution of Eq. (4.8) for $(x + x_1) > 0$ is then

$$y(z) = \left(\frac{3z}{2}\right)^{1/3} \frac{I_{2/3}(z) + NI_{-2/3}(z)}{I_{-1/3}(z) + NI_{1/3}(z)} , \qquad z = \frac{2}{3}(x + x_1)^{3/2}. \tag{4.14}$$

The constant N can be determined by using again the condition (3.13).

The continuation of Eq. (4.14) into the region $(x + x_1) < 0$ is provided by

$$\begin{aligned} z^{1/3}I_{1/3}(z) &= -\zeta^{1/3}J_{1/3}(\zeta), \qquad & z^{1/3}I_{-1/3}(z) &= \zeta^{1/3}J_{-1/3}(\zeta), \\ z^{2/3}I_{2/3}(z) &= \zeta^{2/3}J_{2/3}(\zeta), & z^{2/3}I_{-2/3}(z) &= \zeta^{2/3}J_{-2/3}(\zeta), \end{aligned} \tag{4.15}$$

where

$$\zeta = ze^{-3\pi i/2} = \tfrac{2}{3}[-(x + x_1)]^{3/2}.$$

Consequently Eq. (4.14) becomes, for $(x + x_1) < 0$,

$$y(\zeta) = \left(\frac{3\zeta}{2}\right)^{1/3} \frac{J_{2/3}(\zeta) + NJ_{-2/3}(\zeta)}{J_{-1/3}(\zeta) - NJ_{1/3}(\zeta)} , \qquad \zeta = \frac{2}{3}[-(x + x_1)]^{3/2}. \tag{4.16}$$

To discuss the above solution, we first note that the inviscid solution in this transonic region, to the first order approximation within the $O(\alpha^{2/3})$, is

$$y^2 = x, \tag{4.17}$$

which has two branches for $x > 0$ and gives no solution for $x < 0$. Now before we determine the value of N for the corresponding viscous solution, we may also note that the

general solution, given by (4.14) and (4.16), is a semi-transcendental function of x_1 and the second integration constant N. It can be shown, from the property of $I_\nu(z)$ at large z, that in Eq. (4.14)

$$y \to x^{1/2} \quad \text{as} \quad x \to \infty \quad \text{when} \quad N > -1,$$

$$y \to -x^{1/2} \quad \text{as} \quad x \to \infty \quad \text{when} \quad N = -1,$$

and y has a simple pole at a certain finite z for $N < -1$ (which is, of course, of no physical significance). This result shows that the viscous solutions tend to their respective inviscid values for x large in a manner which implies again that ($w = \beta^{-1/2}$, $\eta = \infty$) is a nodal point (admitting more than one value of N) while ($w = 0$, $\eta = \infty$) is a saddle point (admitting only one value of N). However, for $x + x_1 < 0$, Eq. (4.16) shows obviously that $y(\zeta)$ has an infinite number of isolated simple poles at $\zeta = \zeta_n$ where the denominator vanishes. Since the properties of the solution curves in the (w, V) phase space exhibit no such singularities, the solution (4.16), therefore, represents a good approximation to the real flow only for ζ lying in the interval $0 \leq \zeta \leq \zeta_1 - \delta < \zeta_1$, where ζ_1 is the first pole and δ is a positive number, appropriately chosen such that $y(\zeta_1 - \delta)$ is not yet too large to void our approximation (4.2).

Having obtained the first order solution $w^{(1)}(\xi) = 2^{1/3}(\gamma + 1)^{-1/3} y$, the second order equation (4.7) can be then integrated to yield

$$w^{(2)}(\xi) = e^{-2\varphi(\xi)} \int \psi(\xi) e^{2\varphi(\xi)}\, d\xi, \tag{4.18}$$

where

$$\varphi(\xi) = \int w^{(1)}(\xi)\, d\xi \qquad \text{and} \qquad \psi(\xi) = [w^{(1)}(\xi)]^3 - (1 + \beta)\varphi(\xi) + \text{const.}$$

It is obvious that $w^{(2)}(\xi)$ is bounded wherever $w^{(1)}(\xi)$ is bounded. Consequently, the approximation is good even if we only take the first two terms in (4.2) and (4.3).

In order to obtain some numerical results, we first determine the value N in Eqs. (4.14), (4.16) by using conditions (3.13) to obtain

$$\begin{aligned} N &= -\,0.585 \text{ for the supersonic branch,} \\ N &= -\,1 \qquad \text{for the subsonic branch.} \end{aligned} \tag{4.19}$$

With these values of x_1 and N (see 4.10 and 4.19), the solutions are plotted in Fig. 6 (by using tables, Ref. [8]) from which several interesting results can be deduced as follows:

(i) For the supersonic branch, the transonic solution starts from point A (see Fig. 6) with the coordinates

$$w_1 = 1 + 1.6[(\gamma + 1)/2]^{-1/3}\alpha^{1/3}, \qquad \eta_1 \doteq 2.9[(\gamma + 1)/2]^{1/3}\alpha^{2/3}. \tag{4.20}$$

After the solution curve passes through a point of inflection G and then crosses the line $y = 0$ ($w = 1$) at point B (see also Fig. 4) where

$$\eta_B \doteq 1.02[(\gamma + 1)/2]^{1/3}\alpha^{2/3}(\sim V_B \doteq -1.05[(\gamma + 1)/2]^{-2/3}\alpha^{-1/3},$$

$$(dV/dw)_B \doteq -0.95[(\gamma + 1)/2]^{-1/3}\alpha^{-2/3}),$$

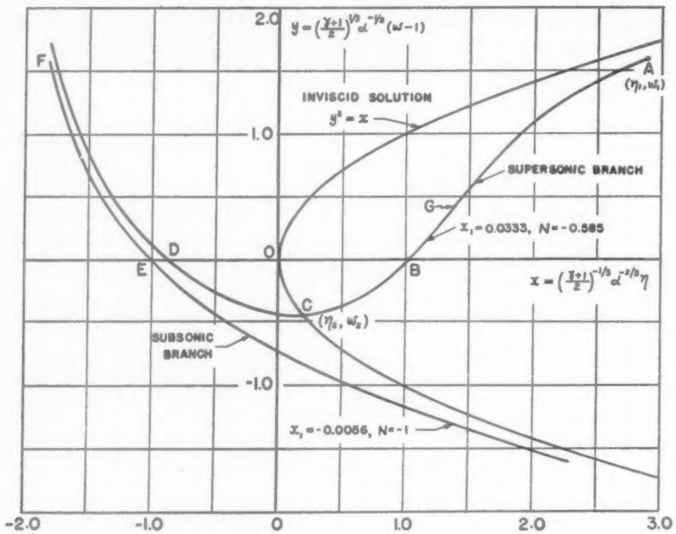

FIG. 6. The flow velocity in the transonic region.

it reaches a minimum when it intercepts the curve $y^2 = x$ at C where

$$w_2 \doteq 1 - 0.45[(\gamma + 1)/2]^{-1/3}\alpha^{1/3}, \qquad \eta_2 \doteq 0.2[(\gamma + 1)/2]^{1/3}\alpha^{2/3}. \tag{4.21}$$

It then increases from $w = w_2$ to $w = 1$ at point D where

$$\eta_D \doteq -0.88\left(\frac{\gamma + 1}{2}\right)^{1/3}\alpha^{2/3},$$

$$\left(\sim V_D \doteq 0.91\left(\frac{\gamma + 1}{2}\right)^{-2/3}\alpha^{-1/3}, \qquad \frac{dV}{dw} \doteq 1.1\left(\frac{\gamma + 1}{2}\right)^{-1/3}\alpha^{-2/3}\right).$$

That is, there is an expansion wave following the cylindrical shock.

(ii) The thickness of the cylindrical shock, as defined in Sec. 2.2d, is

$$\Delta\eta = \eta_1 - \eta_2 \doteq 2.7\left(\frac{\gamma + 1}{2}\right)^{1/3}\alpha^{2/3}, \qquad (\text{in } r\text{-space, } \Delta r \doteq r_1 \Delta\eta), \tag{4.22}$$

across which the velocity has a total variation

$$\Delta w = w_1 - w_2 = 2.05[(\gamma + 1)/2]^{-1/3}\alpha^{1/3}; \tag{4.23}$$

moreover,

$$w_1 w_2 = 1 + 1.15[(\gamma + 1)/2]^{-1/3}\alpha^{1/3}. \tag{4.24}$$

Combining Eqs. (4.22) and (4.23), we obtain

$$(\Delta w)(\Delta\eta) = 5.5\alpha. \tag{4.25}$$

Comparing these results with those of a plane normal shock (e.g. Ref. [10]), we note first that the plane shock strength ($\sim \Delta w$) is quite arbitrary while for a cylindrical shock,

$\Delta w \sim 0(\alpha^{1/3})$. The expression for shock thickness (Eq. 4.22) shows that $\Delta\eta \sim 0(\alpha^{2/3})$, although, after combining with Δw, the expression (4.25) agrees with that of a plane shock [10] within the order of magnitude. The result (4.25) differs, however, from Levey's result for the diffuse shock in a source flow (Ref. [5], Eq. 4.9), in which he explains the discrepancy as due to some degree of choice of the definition of the shock thickness. Our result also indicates that the maximum velocity gradient inside a cylindrical shock is of order $\alpha^{-1/3}$, (in contrast to Levey's result: $0(\alpha^{-1})$), while for a plane shock, the maximum gradient is of order $(\Delta w)^2 \alpha^{-1}$ [10], which reduces to $0(\alpha^{-1/3})$ if $\Delta w \sim 0(\alpha^{1/3})$. The expression (4.24) differs from the Prandtl relation of a plane shock by a term of $0(\alpha^{1/3})$ which here agrees with Levey's result (Ref. [5], Eq. 4.13). The differences between the present results of cylindrical flow and those of one dimensional plane shock can perhaps be realized by visualizing that the viscous forces exerting on the surfaces $rd\theta$ and dr of a fluid element are indeed of quite different nature from those exerting in plane shock flow, since in the former case, the normal stress acting on the surface dr will have a component in the radial direction.

(iii) On the subsonic branch, w is a monotonically decreasing function of η. $w = 1$ at point E (see Figs. 6 and 4) where $\eta_E \doteq -[(\gamma + 1)/2]^{1/3} \alpha^{2/3}$ and the velocity gradient $(dw/d\eta)_E = -V_E = -[(\gamma + 1)/2]^{-2/3} \alpha^{-1/3}$ (see also Eq. 2.19a), which shows that the solution curve of the subsonic branch passes $w = 1$ (see Fig. 4) slightly above the curve G_2.

(iv) The thermodynamic variables in this flow region can be deduced from Eqs. (4.5), (1.7) and (1.8). That is, in the expansion (4.3) and

$$\begin{aligned} p^*(\xi) &= 1 + \alpha^{1/3}p^{(1)}(\xi) + \alpha^{2/3}p^{(2)}(\xi) + \cdots, \\ \rho^*(\xi) &= 1 + \alpha^{1/3}\rho^{(1)}(\xi) + \alpha^{2/3}\rho^{(2)}(\xi) + \cdots, \end{aligned} \tag{4.26}$$

we have

$$\theta^{(1)}(\xi) = -(\gamma - 1)\left(\frac{\gamma + 1}{2}\right)^{-1/3} y(\xi), \tag{4.27}$$

$$p^{(1)} = \gamma\rho^{(1)} = \frac{\gamma}{\gamma - 1}\,\theta^{(1)}, \tag{4.28}$$

where $y(\xi)$ is given in Eqs. (4.14) and (4.16). The value of $\rho^{(1)}$ and $\theta^{(1)}$ *vs.* η is plotted in Fig. 7. The supersonic branch starts with compression and is then followed by an expansion wave, while the subsonic branch expands continuously. Equation (4.28) simply states that p^*, ρ^* and θ satisfy the isentropic relation up to $0(\alpha^{1/3})$. This implies that the entropy variation, if any, across this region must be of order at least $\alpha^{2/3}$.

It should be pointed out here that Eq. (4.17) fails to be a good approximation for $\eta > O(\alpha^{2/3})$. Consequently the transonic solution, which approaches the asymptote given by Eq. (4.17), cannot be extended beyond the transonic region. It then becomes obvious that the patching of these solutions with the outside solutions must be made at $\eta = O(\alpha^{2/3})$. This behavior of the present solution is further indicated by the above result that the values of the variables given by the outside expansion at $\eta = O(\alpha^{2/3})$ fit right into the transonic similarity rule.

5. The solution in the inner supersonic region. The above transonic solution shows that the supersonic branch flow approaches asymptotically to the subsonic branch

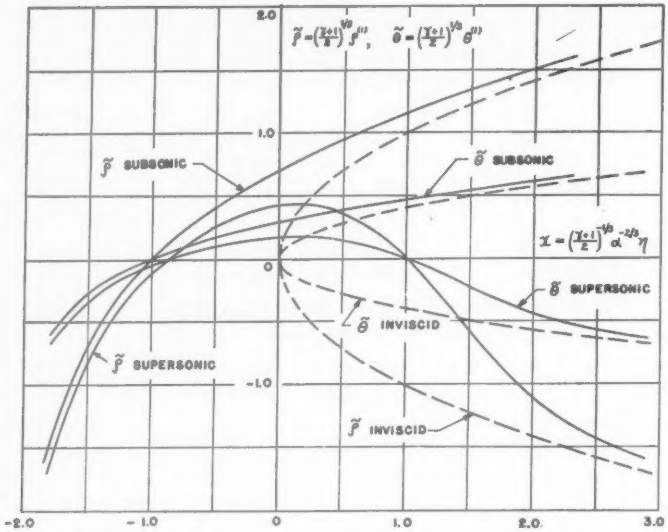

FIG. 7. The thermodynamic variables in the transonic region.

and for $\eta < \eta_E$ both branches have supersonic local speed. We shall proceed to find the solution for $\eta < \eta_E$. Let us consider first the continuation of the subsonic branch. For the sake of convenience, we shall take the point E in Fig. 6 as the boundary such that at E, $w_E = 1$ and

$$\eta_E \doteq -\left(\frac{\gamma + 1}{2}\right)^{1/3} \alpha^{2/3}, \quad V_E \doteq \left(\frac{\gamma + 1}{2}\right)^{-2/3} \alpha^{-1/3}, \quad \left(\frac{dV}{dw}\right)_E \doteq \left(\frac{\gamma + 1}{2}\right)^{-1/3} \alpha^{-2/3}. \tag{5.1}$$

Now rewrite Eq. (2.6) in the following form

$$\frac{dV}{dw} = \frac{1}{\alpha}\left(1 - \frac{1}{w^2}\right) + \frac{1 - \beta w^2}{\alpha w} \frac{1}{V} \cdot \tag{5.2}$$

From the properties of the solution of this equation (see Sec. 2.2) we recall that for $\eta < \eta_E$, $V > V_E$, that is, in this region V is at least of $0(\alpha^{-1/3})$. It then seems convenient to adopt an iteration method to approximate the solution for $\eta < \eta_E$. As the first iteration, we neglect the second term on the right hand side of (5.2) and integrate, using (5.1), to obtain

$$V = \frac{(w - 1)^2}{\alpha w} + \left(\frac{\gamma + 1}{2}\right)^{-2/3} \alpha^{-1/3}. \tag{5.3}$$

Substituting this value of V into the right hand side of Eq. (5.2) and integrating twice, using again Eq. (5.1), we obtain, as the second iteration, the following asymptotic result, for α small, valid for $w > 1$,

$$\eta = \eta_E - \left(\frac{\gamma + 1}{2}\right)^{1/3} \alpha^{2/3} \arctan\left[\frac{2(w - 1)}{w + 1}\left(\frac{\gamma + 1}{2}\right)^{1/3} \alpha^{-1/3}\right] + 0(\alpha \log \alpha). \tag{5.4}$$

This solution is in good agreement with the value given by Eq. (4.16) for $\eta_E < \eta < \eta_F$. It can be seen at once that as $w \to \beta^{-1/2}$ (the maximum velocity at which $\theta = 0$), η approaches its smallest value η_m, say, where

$$\eta_m - \eta_E \doteq -1.6[(\gamma + 1)/2]^{1/3}\alpha^{2/3}. \tag{5.5}$$

Thus we see that the flow supposedly terminates itself at a distance of $0(\alpha^{2/3})$ to the inner side of $\eta = 0$, beyond which there is no solution to our present system of equations. If one were to investigate further the possibility that one could still obtain a solution of physical reality for $\eta < \eta_m$, one would face some rather dubious situations. For instance, near $\eta = \eta_m$, the density, temperature and pressure all become so low that the validity of the equation of state for a perfect gas (1.4) is questionable. Besides, the fact that the viscous stresses reach the magnitude of the fluid pressure near $\eta = \eta_m$ sets a likely limit on the applicability of the Navier-Stokes equation (1.1) and also raises a question as to whether Burnett's higher viscous terms (Ref. [11], p. 271) should be employed to overcome the present difficulty. Of course, it would seem plausible to continue our solution further inward by assigning appropriate values to the arbitrary constant C in (2.16). Nevertheless, it is still impossible to bring the flow to $\eta = -\infty$ $(r = 0)$ on account of the singularity that $\rho u \sim r^{-1}$ near $r = 0$ (see Eq. 1.3). To clarify these rather vague points is beyond the scope of this paper, although such clarification is certainly desirable.

6. The entropy variation. We define S to be the specific entropy,

$$T\,dS = Cp\,dT - \rho^{-1}\,dp \tag{6.1}$$

then the energy equation (1.2) can be written as

$$\rho u T \frac{dS}{dr} = \operatorname{div}(\lambda \operatorname{grad} T) + \Phi, \tag{6.2}$$

where Φ is the viscous dissipation function, which in this case is,

$$\Phi = \frac{2}{3}(\mu' + 2\mu)\left[\left(\frac{du}{dr}\right)^2 + \left(\frac{u}{r}\right)^2\right] + \frac{4}{3}(\mu' - \mu)\frac{u}{r}\frac{du}{dr}\,. \tag{6.3}$$

The above definition of S of a fluid element is clearly for an open system since the heat exchange by conduction, and hence a net flow of entropy, occurs with the neighboring elements. Thus Eq. (6.2) merely expresses the energy balance, in terms of S, of a fluid element—a system not isolated, in the thermodynamic sense, from its surroundings. The analysis of formulating the second law of thermodynamics for the fluid flow case by making the system closed has been investigated in some detail by Tolman and Fine [12] and discussed later by Curtiss and Hirschfelder [13] from the point of view of statistical mechanics. Their idea is, in essence, to state that the change ΔS in the entropy of a system should consist not only of the net increase in entropy produced by irreversible processes taking place inside the system, but also of the entropy carried into the system, due to conduction of heat energy, equal to div $[\lambda T^{-1}$ grad $T]$ per unit volume. Following Tolman's notation, we may thus write

$$\rho\left(\frac{DS}{Dt}\right) = \rho\left(\frac{DS}{Dt}\right)_{\text{irr.}} + \operatorname{div}\left(\frac{\lambda}{T}\operatorname{grad} T\right). \tag{6.4}$$

In the present case, Eq. (6.4), after combining with (6.2), becomes

$$\rho u T\left(\frac{dS}{dr}\right)_{\text{irr.}} = \frac{\lambda}{T}(\text{grad } T)^2 + \Phi. \tag{6.5}$$

As we are only interested in the qualitative features of our later results, we simplify these equations by using the assumptions:

$$\mu^* = \mu/\mu_1 = 1, \qquad \mu' = 0, \qquad C_p, C_v = \text{constant}, \qquad \sigma = 3/4. \tag{6.6}$$

Then the nondimensional form of Eqs. (6.2) and (6.5) are respectively

$$-\frac{\theta}{\gamma - 1}\frac{ds}{d\eta} = \frac{4\gamma}{3}\alpha^*\left\{\frac{1}{\gamma - 1}\frac{d^2\theta}{d\eta^2} + \left(\frac{dw}{d\eta}\right)^2 - w\frac{dw}{d\eta} + w^2\right\} \tag{6.7}$$

$$-\frac{\theta}{\gamma - 1}\left(\frac{ds}{d\eta}\right)_{\text{irr.}} = \frac{4\gamma}{3}\alpha^*\left\{\frac{\theta^{-1}}{\gamma - 1}\left(\frac{d\theta}{d\eta}\right)^2 + \left(\frac{dw}{d\eta}\right)^2 - w\frac{dw}{d\eta} + w^2\right\} \tag{6.8}$$

where

$$s = S/C_v\,. \tag{6.9}$$

Though the sign of the terms on the right hand side of (6.7) is in general indefinite, the value of the right hand side terms of (6.8) is, however, positive definite. Therefore $(s)_{irr}$ increases monotonically along the fluid flow, as predicted by the second law for a closed system. Subtracting (6.8) from (6.7), we obtain an equation which can be integrated to yield

$$s_{\text{irr.}} = s + \frac{4\gamma}{3}\alpha^*\frac{d \log \theta}{d\eta}, \tag{6.10}$$

where the constant of integration is so chosen that both s and s_{irr} tend to s_0 as $\eta \to \infty$, s_0 being arbitrary.

In order to see that s of the shock type flow reaches a maximum near $w = 1$, we substitute Eq. (2.2) with $E = 0$ into (6.7) and obtain

$$\frac{\theta}{\gamma - 1}\frac{ds}{d\eta} = \frac{4\gamma}{3}\alpha^* w\left(\frac{d^2w}{d\eta^2} + \frac{dw}{d\eta} - w\right). \tag{6.11}$$

This equation shows that for η outside the transonic region, the variation in s is at most of $0(\alpha)$. Within the transonic region, $d^2w/d\eta^2$, being of $0(\alpha^{-1})$, overwhelms the rest of the terms in the bracket and hence (6.11) reduces to

$$\frac{\theta}{\gamma - 1}\frac{ds}{d\eta} = \frac{4\gamma}{3}\alpha^* w\frac{d^2w}{d\eta^2}(1 + 0(\alpha^{2/3})). \tag{6.12}$$

It then follows that s assumes its maximum value at the point where the curvature of the $w = w(\eta)$ curve vanishes ($d^2w/d\eta^2 = 0$ at point G in Fig. 6 and at this point $d^2s/d\eta^2$ is less than zero). However, from (6.10), the quantity $s + (4\gamma\alpha^*/3)d \log \theta/d\eta$ does not have an extremum in the entire flow region. The above result is very much the same as that of a plane shock, as it can be shown that the velocity has a point of inflection at sonic speed where the entropy is also a maximum.

Integrating Eq. (6.11) with the aid of Eqs. (3.1) and (3.2) under condition (6.6), we obtain

$$s - s_0 = \log [\theta(wr^*)^{(\gamma-1)}] = \log \theta + (\gamma - 1)(\log w + \eta), \tag{6.13}$$

where

$$s_0 = \log (p_0\rho_0^{-\gamma}),$$

so that $s \rightarrow s_0$ as $r \rightarrow \infty$. This equation is actually the definition of s usually given for a perfect gas. Substitution of the solution (3.12) into (6.13) shows that

$$\Delta s \sim 0(\alpha) \qquad \text{for} \qquad \eta > 0(1). \tag{6.14}$$

Within and around the transonic region, we substitute the solution (4.1)-(4.3) into (6.13) and simplify the expansion, then we obtain

$$s - s_0 = \alpha^{2/3}(\gamma - 1)[(\gamma + 1)/2]^{1/3}(x - y^2) + 0(\alpha), \tag{6.15}$$

where x and y are defined by Eq. (4.9). The value of $y = y(x)$ is given by Eqs. (4.14), (4.16) and also plotted in Fig. 6. Equation (6.15) is consistent with the fact that $s = s_0 =$ constant along the inviscid solution $y^2 = x$. The variation in s along supersonic and subsonic branches of our solution follows directly from the data shown in Fig. 6. The result is plotted in Fig. 8. As η decreases along the supersonic branch, the entropy s

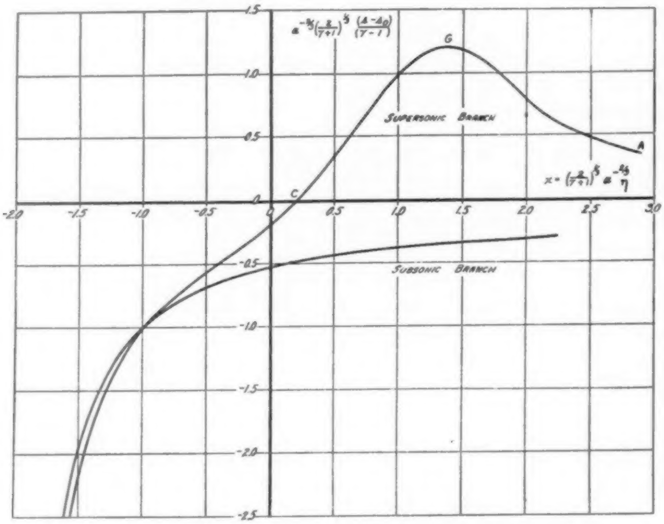

FIG. 8. The entropy variation across the transonic region.

first increases until it reaches the maximum $s_0 + 1.2\ \alpha^{2/3}\ (\gamma - 1)\ [(\gamma + 1)/2]^{1/3}$ at point G, then decreases and later assumes once again the value s_0 (the value of s at $\eta = \infty$) at point C where $w^{(1)}$ is minimum. After that s decreases rapidly with further decrease in η and eventually tend to $-\infty$ as the flow solution terminates. On the subsonic branch, s decreases monotonically with decreasing η. However, by substituting Eqs. (6.15) and (4.3) into (6.10), it can easily be shown that $(s_{\text{irr.}})$ increases monotonically

with decreasing η and the variation in $(s_{irr.})$ is of order $0(\alpha)$. Consequently, the result that $s \to -\infty$ as $\eta \to \eta_{min}$ can be explained by visualizing from Eq. (6.10) that $d(\log \theta)/d\eta$ decreases beyond all bounds as $\eta \to \eta_{min}$. Physically, this probably implies that the flow is rather far from its equilibrium condition due to the large velocity gradient, inducing a rapid decrease in temperature, which even the important heat conduction in this region cannot compensate.

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ON THE STABILITY OF THE SPHERICAL SHAPE OF A VAPOR CAVITY IN A LIQUID*

BY

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1. Introduction. It has been shown by G. I. Taylor [1] that a plane interface between two fluids of different densities in accelerated motion is stable or unstable according as the acceleration is directed from the heavier to the lighter fluid, or conversely. This stability analysis is limited to small amplitude perturbations of the plane interface; and it is found that a small distortion of the interface begins to grow exponentially with time in the unstable situation and to decrease exponentially in the stable situation. While experimental observations agree well with the theory in the small amplitude limit for which the theory is valid, it is known that there are significant deviations in the rate of growth of distortions in the unstable case when their amplitudes become appreciable [2].

It is of interest to consider the analogous stability problem for the case of a spherical interface between two immiscible fluids of different densities in accelerated motion. For perturbations in the spherical interface of small amplitude, it may be shown [3] that the stability criterion deduced by Taylor for the plane case is subject to important modifications.

The stability problem in the spherical case may be formulated as follows. A fluid of density ρ_1 is contained within a sphere of radius R; a fluid of density ρ_2 occupies the region exterior to this sphere. The fluids are supposed to be immiscible, incompressible and nonviscous. The equation of motion for the interface radius as a function of time, $R(t)$, is readily determined under the assumption that the initial and boundary conditions are spherically symmetric. If the interface is distorted from the surface of a sphere of radius R to a surface with radius vector of magnitude r_s, then one may write

$$r_s = R + \sum a_n Y_n , \tag{1}$$

where Y_n is a spherical harmonic of degree n and the a_n's are functions of the time to be determined. The stability of the spherical interface may be established by considering whether interface distortions of small amplitude grow or diminish. More precisely, it is assumed that

$$| a_n(t) | \ll R(t),$$

and that terms of order higher than the first in a_n and da_n/dt are negligible. In such a linearized perturbation theory, the a_n's are independent of each other, and it may be shown [3] that they satisfy the following differential equation

$$\frac{d^2a_n}{dt^2} + \frac{3}{R}\frac{dR}{dt}\frac{da_n}{dt} - Aa_n = 0 \tag{2}$$

with

$$A = \frac{[n(n-1)\rho_2 - (n+1)(n+2)\rho_1]d^2R/dt^2 - (n-1)n(n+1)(n+2)\sigma/R^2}{[n\rho_2 + (n+1)\rho_1]R}, \tag{3}$$

where σ is the surface tension constant.

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The spherical stability problem has application to the behavior of growing or collapsing gas bubbles in a liquid. Penney and Price [4] have carried out a numerical solution of the stability equation (2) for $n = 2$ for the case of a pulsating gas bubble in water with an internal pressure, p_i, in the bubble given by

$$p_i R^{3\gamma} = \text{const.}$$

and with a constant pressure, p_0, in the liquid at a distance from the bubble. In their computations surface tension is neglected. The numerical solution showed that the distortion amplitude a_2 is much larger when the bubble is near its minimum radius than elsewhere. The problem to be considered here is a cavity for which the internal pressure, p_i, is constant, in a liquid at constant external pressure p_0. These are approximately the conditions in a vapor cavitation bubble in a liquid. An analytic solution for the stability equation may be found under these conditions.

2. Solution of the stability problem. For a vapor cavity in a liquid, the vapor density ρ_1 may be neglected in comparison with the liquid density ρ_2. The quantity A of Eq. (3) then becomes

$$A = \frac{(n-1)}{R}\frac{d^2R}{dt^2} - (n-1)(n+1)(n+2)\frac{\sigma}{\rho R^3}, \tag{4}$$

where $\rho = \rho_2$ is the liquid density. The equation of motion of the undisturbed interface [3] is

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 = \frac{p_i - p_0 - 2\sigma/R}{\rho}. \tag{5}$$

Equation (5) may also be written as

$$\frac{d}{dt}\left[R^3\left(\frac{dR}{dt}\right)^2\right] = 2R^2\frac{dR}{dt}\left[\frac{p_i - p_0 - 2\sigma/R}{\rho}\right]$$

which integrates, when $p_i - p_0$ is a constant, to give

$$2\pi\rho\left[R^3\left(\frac{dR}{dt}\right)^2 - R_0^3\left(\frac{dR_0}{dt}\right)^2\right] = \frac{4\pi}{3}(p_i - p_0)(R^3 - R_0^3) - 4\pi\sigma(R^2 - R_0^2), \tag{6}$$

where R_0 is the cavity radius and dR_0/dt is its radial velocity at $t = t_0$. This integral of Eq. (5) is to be recognized as the energy integral of the system.

The general features of the asymptotic behavior of the distortion amplitude a_n may be made evident in a straightforward way. For the case of an expanding bubble, Eq. (6) gives

$$\left(\frac{dR}{dt}\right)^2 \sim \frac{2P}{3\rho}, \qquad R \to \infty \tag{7}$$

with $P = p_i - p_0 > 0$, and it then follows from Eqs. (5) and (2) that

$$a_n \to \text{const.}, \qquad R \to \infty. \tag{8}$$

For the case of a collapsing bubble, it is convenient to transform Eq. (2) by the substitution

$$a_n = \left(\frac{R_0}{R}\right)^{3/2} b_n \tag{9}$$

into

$$\frac{d^2b_n}{dt^2} - G(t)b_n = 0 \tag{10}$$

with

$$G(t) = \frac{3}{2}\frac{d}{dt}\left[\frac{1}{R}\frac{dR}{dt}\right] + \frac{9}{4}\left[\frac{1}{R}\frac{dR}{dt}\right]^2 + A$$

$$= \frac{3}{4}\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{(n+1/2)}{R}\frac{d^2R}{dt^2} - (n-1)(n+1)(n+2)\frac{\sigma}{\rho R^3}. \tag{11}$$

From Eq. (6), one has

$$\left[\frac{dR}{dt}\right]^2 = \frac{R_0^3}{R^3}\left[\left(\frac{dR_0}{dt}\right)^2 + \frac{2p}{3\rho} + \frac{2\sigma}{\rho R_0}\right] + 0\left[\frac{1}{R}\right] \tag{12}$$

where $p = p_0 - p_i > 0$ for this case. The radial acceleration, d^2R/dt^2, is determined by Eq. (5); and the function $G(t)$ is found to be

$$G(t) \sim -\frac{3n}{2}\frac{R_0^3}{R^5}\left[\left(\frac{dR_0}{dt}\right)^2 + \frac{2p}{3\rho} + \frac{2\sigma}{\rho R_0}\right], \qquad R \to 0, \tag{13}$$

except for smaller terms. It is evident that

$$G(t) \sim -\frac{nc^2}{R^5}, \tag{14}$$

where c is a real constant. One may now write a W.K.B. approximation to the solution of Eq. (10) for small R in the form

$$b_n \sim [G(t)]^{-1/4} \exp\left\{\pm\int^t [G(t')]^{1/2}\, dt'\right\} \sim R^{5/4} \exp\left\{\pm icn^{1/2}\int^t R^{-5/2}\, dt'\right\}. \tag{15}$$

The distortion amplitude a_n is then given by

$$a_n \sim R^{-1/4} \exp\left\{\pm icn^{1/2}\int^t R^{-5/2}\, dt'\right\}, \qquad R \to 0; \tag{16}$$

so that a_n increases like $R^{-1/4}$ in amplitude and oscillates with increasing frequency as $R \to 0$. This behavior has been found by Birkhoff [5] by a different procedure. It is of interest that the instability found by Birkhoff near $R = 0$ is qualitatively unaffected by surface tension.

The question remains over what range of R is the linearized perturbation theory for the distortion amplitude*, a, valid or consistent. The following problem will therefore be solved. A spherical cavity with radius R_0 at $t = 0$ expands, or collapses, from rest, $dR_0/dt = 0$, under a constant pressure difference; at $t = 0$, the cavity is supposed to have a distortion of small amplitude a_0 , and the subsequent behavior of a for any R is to be determined. Complete solutions for this problem are readily found when surface tension is neglected and these solutions are given first. The effects of surface tension will then be illustrated by some special solutions.

(i) *Expanding cavity; no surface tension*

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*The subscript n for the distortion amplitude will be omitted in the following.

With no surface tension, the stability equation to be solved simplifies to

$$\frac{d^2a}{dt^2} + \frac{3}{R}\frac{dR}{dt}\frac{da}{dt} - \frac{(n-1)}{R}\frac{d^2R}{dt^2}a = 0; \tag{17}$$

one finds from Eq. (6) that

$$\left[\frac{dR}{dt}\right]^2 = \frac{2P}{3\rho}\left[1 - \frac{R_0^3}{R^3}\right], \tag{18}$$

where $P = p_i - p_0 > 0$; and from Eq. (5) that

$$\frac{d^2R}{dt^2} = \frac{P}{\rho}\frac{R_0^3}{R^4}. \tag{19}$$

If the independent variable in Eq. (17) is changed from t to the volume ratio

$$x = \frac{R_0^3}{R^3}, \qquad 0 \le x \le 1; \tag{20}$$

there results

$$x(1-x)\frac{d^2a}{dx^2} + \left[\frac{1}{3} - \frac{5}{6}x\right]\frac{da}{dx} - \frac{(n-1)}{6}a = 0. \tag{21}$$

Equation (21) will be recognized as the differential equation for the hypergeometric function $F(\alpha, \beta; \gamma; x)$ where the parameters have the values

$$\begin{aligned} \alpha &= \frac{-1 + i(24n - 25)^{1/2}}{12} = -\frac{1}{12} + i\delta; \\ \beta &= \frac{-1 - i(24n - 25)^{1/2}}{12} = \alpha^*; \\ \gamma &= \frac{1}{3}. \end{aligned} \tag{22}$$

It is convenient to take the general solution of Eq. (21) in the form [6]

$$a = AF(\alpha, \beta; 1/2; 1 - x) + B(1 - x)^{1/2}F(-\alpha + 1/3, -\beta + 1/3; 3/2; 1 - x) \tag{23}$$

where A and B are constants to be fixed by the initial conditions. If $a = a_0$ at $t = 0$ or $R = R_0$, then one has

$$A = a_0 \tag{24}$$

from Eq. (23). Similarly, if the initial velocity amplitude for the distortion is v_0, then

$$\begin{aligned} v_0 = \left[\frac{da}{dt}\right]_{t=0} &= \lim_{x\to 1}\left[\frac{da}{dx}\frac{dx}{dt}\right], \\ &= \frac{3B}{2R_0}\left[\frac{2P}{3\rho}\right]^{1/2}, \end{aligned} \tag{25}$$

which fixes the constant B. The quantity $(2P/3\rho)^{1/2}$ is a characteristic velocity and R_0 is a characteristic length for the system, and it is convenient to describe the initial velocity amplitude in terms of the length

$$l_0 = \frac{v_0 R_0}{(2P/3\rho)^{1/2}}, \tag{26}$$

and Eq. (25) becomes

$$B = 2l_0/3. \tag{27}$$

The limiting value of a as $R \to \infty$, or $x \to 0$, is readily determined from Eq. (23). One has

$$a(R \to \infty) = a_\infty = \pi^{1/2}\Gamma(2/3)\left[\frac{a_0}{|\,\Gamma(1/2 - \alpha)\,|^2} + \frac{l_0}{3\,|\,\Gamma(7/6 + \alpha)\,|^2}\right]$$

so that for large n

$$a_\infty \sim \frac{1}{2}\pi^{-1/2}e^{\pi\delta}\Gamma(2/3)\left[a_0\delta^{-1/6} + \frac{l_0}{3}\,\delta^{-7/6}\right], \tag{28}$$

where from Eq. (22)

$$\delta = \frac{(24n - 25)^{1/2}}{12}.$$

Figure 1 shows the variation of a/a_0 with R_0/R for various values of n for the case in which the initial velocity amplitude is zero, $l_0 = 0$; Figure 2 shows the variation of a/a_0 for the case in which the initial velocity amplitude is different from zero. Of greater significance is the ratio of the distortion amplitude a to the mean bubble radius R; the

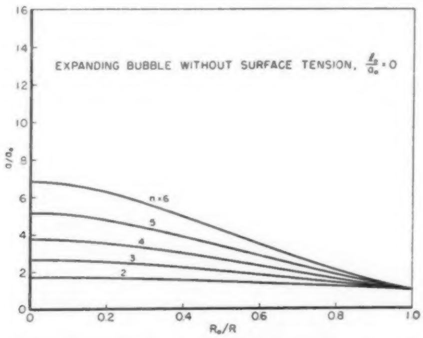

FIG. 1. The ratio of the distortion amplitude a to its initial value a_0 is shown for an expanding vapor cavity as a function of R_0/R where R_0 is the initial cavity radius and R its radius at later times. The distortion of the spherical interface is a_nY_n where Y_n is a spherical harmonic of order n. The initial velocity of the distortion is zero.

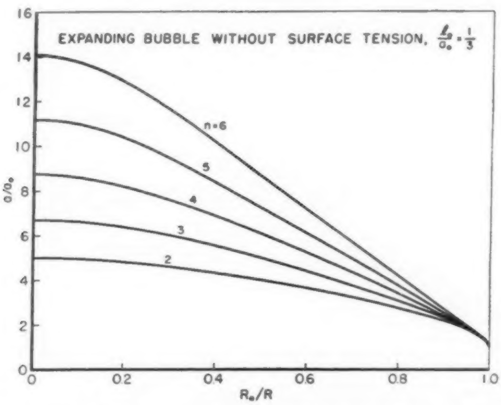

FIG. 2. The distortion amplitude is shown as a function of cavity radius for a non-zero initial distortion velocity.

behavior of (a/a_0) (R_0/R) is shown as a function of R_0/R in Fig. 3 for the case in which the initial velocity amplitude is zero, and in Fig. 4 for the case in which the initial velocity amplitude is different from zero.

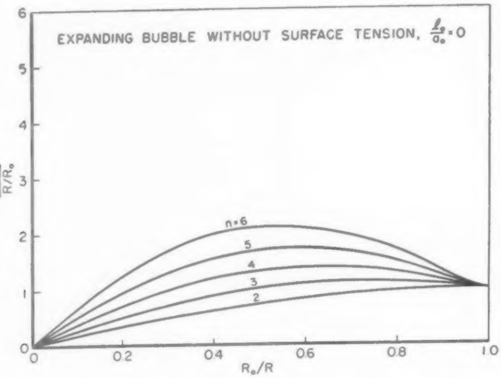

FIG. 3. The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_0/R_0) is shown as a function of R_0/R for the case shown in Fig. 1.

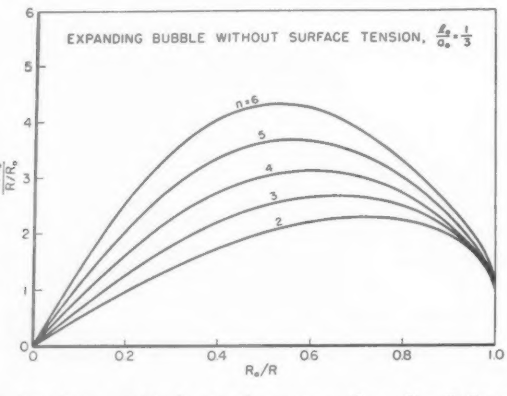

FIG. 4. The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_0/R_0) is shown as a function of R_0/R when the initial distortion amplitude velocity is non-zero.

(ii) *Collapsing cavity; no surface tension*

Equation (21) is also applicable to this case. It is convenient, however, to write the solution in the form [6]

$$a = Ax^{-\alpha}F(\alpha, \alpha + 2/3; 1/2; 1 - 1/x)$$
$$+ Bx^{-\alpha}(1 - 1/x)^{1/2}F(-\beta + 1/3, -\beta + 1; 3/2; 1 - 1/x), \qquad 1 \leq x \leq \infty. \tag{29}$$

Equation (18) is written in the form

$$\left[\frac{dR}{dt}\right]^2 = \frac{2p}{3\rho}\left[\frac{R_0^3}{R^3} - 1\right],$$

where now $p = p_0 - p_i > 0$. The constants A and B are found as in the previous case in terms of the initial distortion amplitude a_0 and the initial velocity amplitude v_0 :

$$A = a_0 ;$$

$$B = \frac{2}{3} v_0 \frac{R_0}{(2p/3\rho)^{1/2}} = \frac{2}{3} l_0 .$$

The solution may, of course, be written in a variety of forms. In place of Eq. (29) one may write, for example,

$$a = A'y^{\alpha}F(\alpha, \alpha + 2/3; 2\alpha + 7/6; y) + B'y^{-\alpha-1/6}F(-\alpha + 1/2, -\alpha - 1/6; -2\alpha + 5/6; y), \tag{30}$$

where now

$$y = \frac{1}{x} = \frac{R^3}{R_0^3}, \qquad 0 \leq y \leq 1. \tag{31}$$

A' and B' are linear combinations of a_0 and l_0 which will not be written explicitly here. From Eq. (30), one finds in the neighborhood of $y = 0$ that

$$a \approx A'y^{-1/12+i\delta} + B'y^{-1/12-i\delta},$$

or

$$a \approx \text{const} \times R^{-1/4}, \tag{32}$$

which is the singularity noted by Birkhoff.

The variation of the distortion amplitude with mean bubble radius is shown in Fig. 5 for $n = 3$ and in Fig. 6 for $n = 6$. The quantity of significance is the ratio of a to R; therefore, the variation of (a/a_0) (R_0/R) with R/R_0 is shown in Fig. 7 for $n = 3$ and in Fig. 8 for $n = 6$.

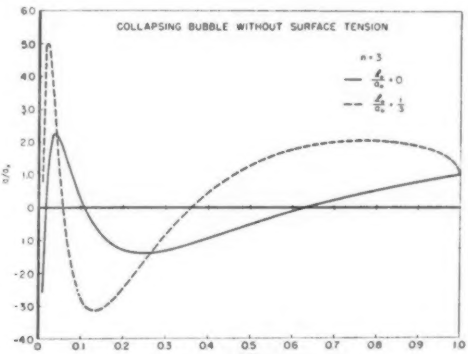

FIG. 5. The ratio of the distortion amplitude a to its initial value a_0 is shown for a collapsing vapor cavity as a function of R/R_0. The case shown is for a distortion described by a spherical harmonic of order $n = 3$.

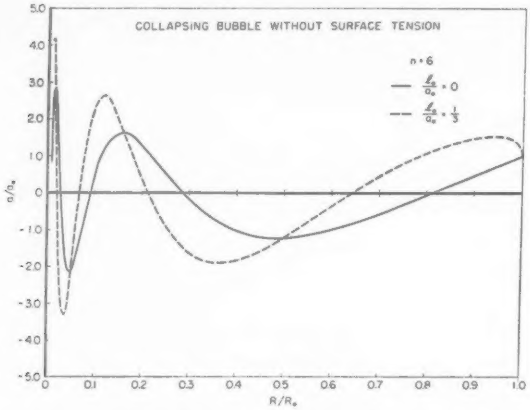

FIG. 6. The distortion amplitude is shown as a function of cavity radius for the case $n = 6$.

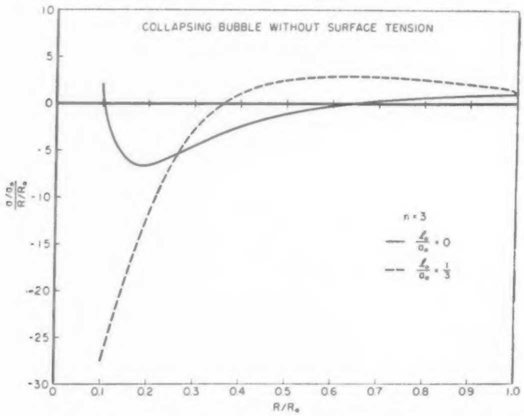

FIG. 7. The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_0/R_0) is shown as a function of R/R_0 for $n = 3$.

(iii) *Expanding cavity with surface tension*

For a bubble expanding from rest, $dR_0/dt = 0$, one has from Eq. (6), if surface tension is included,

$$\left[\frac{dR}{dt}\right]^2 = \frac{2P}{3\rho}\left[1 - \frac{R_0^3}{R^3}\right] - \frac{2\sigma}{\rho R}\left[1 - \frac{R_0^2}{R^2}\right], \tag{33}$$

where $P = p_i - p_0 > 0$. The radial acceleration is determined by the relation

$$R\,\frac{d^2R}{dt^2} + \frac{3}{2}\left[\frac{dR}{dt}\right]^2 = \frac{P - 2\sigma/R}{\rho}.$$

If the stability equation (2) is written in terms of the independent variable

$$z = \frac{R_0}{R} \tag{34}$$

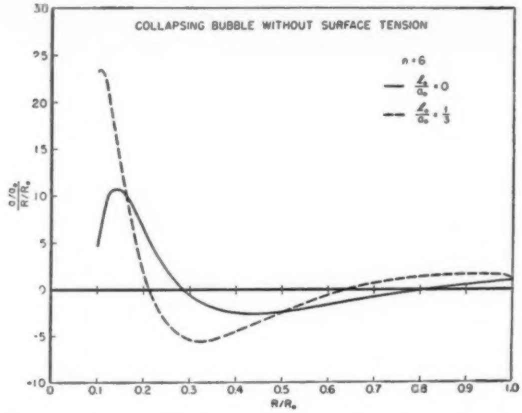

FIG. 8. The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_0/R_0) is shown as a function of R/R_0 for $n = 6$.

it becomes

$$z\left\{\frac{2}{3} - kz + (k - 2/3)z^3\right\}\frac{d^2a}{dz^2} - \left\{\frac{2}{3} - \frac{k}{2}z - \frac{1}{3}\left(\frac{3k}{2} - 1\right)z^3\right\}\frac{da}{dz}$$
$$- (n-1)\left\{\frac{k}{2}\left[1 - (n+1)(n+2)\right] + \left(1 - \frac{3k}{2}\right)z^2\right\}a = 0, \qquad (35)$$

where

$$k = \frac{2\sigma}{R_0 P} \qquad (36)$$

so that k is the ratio of the initial value of the surface tension to the static pressure difference between the inside of the bubble and the liquid. Equation (35) has a neat solution for the special value of $k = 2/3$ in which case it reduces to the hypergeometric differential equation. This value of k is reasonable for vapor bubbles growing in superheated water where it is effectively slightly smaller than unity [7]. A convenient form for the solution is

$$a = AF(\alpha, \beta; 1/2; 1 - z) + B(1 - z)^{1/2}F(\alpha + 1/2, \beta + 1/2; 3/2; 1 - z) \qquad (37)$$

where now

$$\alpha = -\frac{3}{4} + \frac{(9 + 16N)^{1/2}}{4}; \qquad (38a)$$

$$\beta = -\frac{3}{4} - \frac{(9 + 16N)^{1/2}}{4}; \qquad (38b)$$

and

$$N = \frac{(n-1)}{2}(n^2 + 3n + 1). \qquad (39)$$

If a_0 is the initial distortion amplitude and v_0 the initial distortion velocity amplitude, then one finds

$$A = a_0,$$

and

$$B = 2l_0 ,$$

where

$$l_0 = \frac{R_0}{(2P/3\rho)^{1/2}} v_0 .$$

The variation of a/a_0 with R_0/R when $k = 2/3$ is shown in Fig. 9 for $n = 2$ and 3. In Fig. 10, the variation of (a/a_0) (R_0/R) with R_0/R is shown for these same values of n.

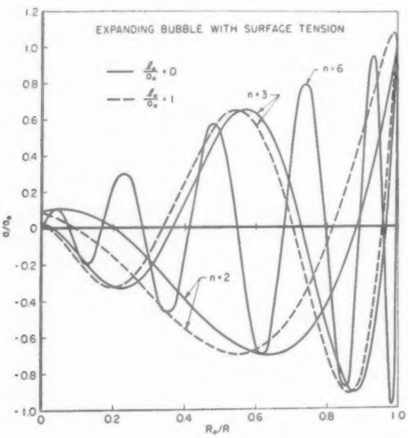

FIG. 9. The distortion amplitude a relative to its initial value a_0 is shown for an expanding cavity as a function of R_0/R for the case in which the effect of surface tension is included. For $n = 6$ the curve with $l_0/a_0 = 1$ is not shown since it lies quite close to the curve $l_0/a_0 = 0$.

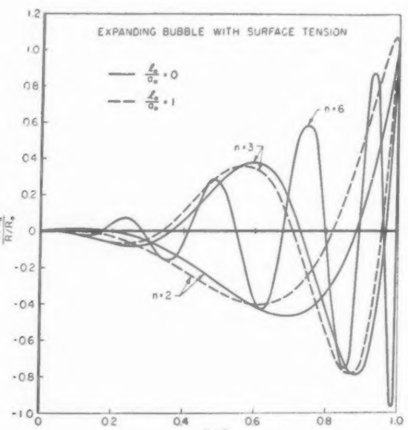

FIG. 10. The ratio of the distortion amplitude a to the mean cavity radius (in units of a_0/R_0) is shown as a function of R_0/R for the case in which the effect of surface tension is included. For $n = 6$ the curves $l_0/a_0 = 1$ and $l_0/a_0 = 0$ are very near each other.

When these curves are compared with Figs. 1 and 2, or Figs. 3 and 4, the stabilizing effect of surface tension is evident. It is also of interest to observe that, when surface tension is included, a/a_0 changes sign as R increases.

(iv) *Cavity collapse under surface tension alone*

An additional case of special analytic simplicity occurs for $p_i = p_0$ so that the cavity collapses under the influence of surface tension alone. If the independent variable in the stability equation for a is changed from t to the area ratio

$$u = \frac{R^2}{R_0^2}, \tag{40}$$

the solution of the resulting differential equation is readily found to be

$$a = u^m[AF(\alpha, \beta; 1/2; 1 - u) + B(1 - u)^{1/2}F(\alpha + 1/2, \beta + 1/2; 3/2; 1 - u)], \tag{41}$$

where m has the value

$$m = \frac{-1 + i(24n - 25)^{1/2}}{8}, \tag{42}$$

and F is the hypergeometric function with parameters α and β determined by the relations

$$\alpha\beta = \frac{m}{2} - \frac{(n - 1)}{8}(n^2 + 3n + 4);$$

$$\alpha + \beta = 2m + \frac{3}{4}.$$

If the initial distortion amplitude is a_0 and the initial distortion velocity amplitude is v_0, then

$$A = a_0,$$

and

$$B = -L_0,$$

where

$$L_0 = v_0 \frac{R_0}{(2\sigma/\rho R_0)^{1/2}}.$$

It is evident from Eq. (41) that

$$a \to \text{const} \times R^{-1/4} \qquad \text{as} \qquad R \to 0.$$

3. Conclusion. For an expanding vapor cavity, an initially spherical shape is stable in the sense that the deformation amplitude a remains small compared with R if its initial value a_0 is small compared with the initial cavity radius R_0. The consistency and applicability of the linearized perturbation theory for the distortion amplitude is thus demonstrated. These conclusions from the linearized theory must be qualified for the case in which surface tension is negligible. As is shown graphically in Figs. 3 and 4, a/R as a function of R_0/R has a maximum which increases slowly with n, the order of the spherical harmonic. It follows, therefore, when surface tension is unimportant, that needlelike irregularities in the spherical interface may grow to significant amplitudes. The present linearized theory is inadequate to follow the development of these high

order distortions of the interface. This instability for large n disappears when surface tension is of significance so that no such restriction need be imposed on the applicability of the linearized theory in this case.

For a collapsing vapor cavity, on the other hand, the perturbation theory is valid provided the distortion amplitude is not followed to small cavity radii. If R_0 is the initial radius of the spherical cavity, then the distortion amplitudes remain small so long as $1 \geq R/R_0 \gtrsim 0.2$ where the lower limit is, of course, approximate. The linearized theory is thus applicable over an interesting and important range of cavity radius. As $R \to 0$, the distortion amplitudes oscillate in sign with increasing frequency and increase in magnitude like $R^{-1/4}$. This increase in distortion amplitude as $R^{-1/4}$ is found with and without surface tension. It may be remarked that the linearized perturbation theory for the distortion amplitudes breaks down in a range of radii near that for which the present model of the vapor cavity becomes invalid. It is known [8] that the vapor pressure within a collapsing vapor cavity, such as is encountered in cavitating flow, begins to rise very rapidly as R/R_0 becomes smaller than approximately 0.1.

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NON-LINEAR NETWORK PROBLEMS*

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1. Flow problems. We shall be concerned with connected *networks*. These will be defined as finite *connected* graphs, on which the *boundary* is explicitly specified.

As a *graph*[1], such a network consists of a finite set N of *nodes* (or vertices), $A_1, \cdots, A_n$, certain nodes being joined in pairs by a finite set L of oriented *links* (or branches) $a_1, \cdots, a_r$. Thus the graph is specified by an incidence matrix of n rows and r columns, $\| \epsilon_{kj} \|$, where ϵ_{kj} is $+1$, -1, or 0 according to whether the node A_k is the initial node, the final node, or not incident on the oriented link a_j. It will be assumed that each link a_j joins exactly two nodes, hence we may write $a_j = A_{i(j)}A_{f(j)}$, where $A_{i(j)}$ is the initial node of the link a_j and $A_{f(j)}$ is the final node of the link a_j. This implies that the incidence matrix has just two non-zero entries in each column (one being $+1$ and one -1). It will also be assumed that each node A_k is incident on at least one link. This implies that each row of the incidence matrix has at least one non-zero entry.

Further, a subset ∂N of N, called the *boundary*, is supposed to be specified. This subset ∂N may or may not be empty. If ∂N is not empty then the elements of ∂N are called the *terminals* of the network. Finally, the network is supposed to be *connected*[2] in the usual sense that a graph is said to be connected.

We shall consider first a special class of network problems, which we shall call "flow problems". Whether they concern hydraulic networks or direct (electrical) current networks, flow problems involve two real valued functions: a *potential* function $u(A_k)$ defined on the nodes, and a *current* function $i(a_j)$ defined on the oriented links. In hydraulic networks $u(A_k)$ is the pressure head; in direct current problems, it represents the voltage.**

In network flow problems, leaks are neglected. One thus assumes, at each interior node A_h in $N - \partial N$, *Kirchhoff's node law*

$$\sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0, \qquad h = 1, \cdots, n; \tag{1}$$

where, in view of the definition of the incidence matrix, the summation is effectively

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[1] See W. H. Ingram and C. M. Cramlet [12], J. L. Synge [20], and the books by O. Veblen [7] and D. König [9]. The basic ideas are due to G. Kirchhoff [1] and H. Poincaré [4, 5]. (Numbers in square brackets refer to the bibliography at the end of the paper).

[2] Actually, this assumption plays a very small rôle, but it simplifies the statement of various results.

**Professor W. Prager has kindly drawn our attention to the occurrence of similar flow problems in the mathematical study of the distribution of traffic over a network of roads, see [28].

taken only over the links incident on the node A_h. For physical equilibrium, the currents must also satisfy certain *equilibrium relations*

$$i(a_j) = c_j(\Delta u_j), \qquad j = 1, \cdots, r, \tag{2}$$

where $\Delta u_j = u(A_{i(j)}) - u(A_{f(j)})$, with $A_{i(j)}$ the initial and $A_{f(j)}$ the final node, respectively, of the oriented link a_j.

Physically, the *conductivity* functions $c_j(\Delta u)$ are usually increasing and continuous. In our theorems below, we shall usually assume one or both of these conditions. For reference, we write[3]

$$c_j(\Delta u) \text{ is a strictly increasing function of } \Delta u, \tag{2a}$$

$$c_j(\Delta u) \text{ is a continuous function of } \Delta u. \tag{2b}$$

Thus, in hydraulic problems, it is commonly assumed that

$$c_j(\Delta u) = K_j \cdot \text{sign}\,(\Delta u) \cdot |\,\Delta u\,|^\alpha, \qquad \text{where} \qquad K_j > 0, \qquad \alpha > 0. \tag{2c}$$

(For turbulent flow in pipes, $\alpha = 1.85$ is commonly accepted.) In direct current problems, a *linear* relation

$$i(a_j) = c_j \Delta u_j\,, \tag{2d}$$

is generally used. Since the general case will be considered in Sec. 4, we shall omit the physical condition $c_j > 0$, which corresponds to (2c) with $\alpha = 1$, and gives the classical case treated by Kelvin [2].

In summary, we will assume (1) at all "interior" nodes (i.e., the nodes of $N - \partial N$) and (2) on all links. At each node A_h of the boundary ∂N, the total "influx" ν_h must clearly satisfy

$$\nu_h = \sum_j \epsilon_{hj} i(a_j). \tag{3}$$

Comparing this last equation with (1), we get the necessary condition

$$\sum \nu_h = 0, \tag{3'}$$

summed over ∂N. [This follows because

$$\sum_{h=1}^{n} \sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0,$$

and (1) then implies

$$\sum_{\partial N} \sum_{j=1}^{r} \epsilon_{hj} i(a_j) = 0,$$

which is (3').]

To obtain a "boundary value problem", some condition must be given at each *terminal* A_h in ∂N; for example, one might assume

I. The potential $u(A_h)$ is given, or

II. The "total influx" ν_h at A_h is given.

Because of the obvious analogy with potential theory, we shall refer to a boundary value problem in which a condition of Type I is given at each terminal as a "Dirichlet

[3] The significance of (2a) and (2b) was first stressed by d'Auriac [14] and Duffin [17].

problem". Similarly, if the total influx is specified at each terminal we shall speak of a "Neumann problem". Problems involving both types have been treated in the literature[4]. Still more generally, one can consider "mixed" conditions, of the type (notice that II′ includes II as a special case)

II′. A functional relation $\nu_h = F_h(u)$ is given, where

$$F_h(u) \text{ is a non-increasing, continuous function of } u. \tag{4}$$

(In heat flow problems, this would correspond to a linear or non-linear "law of cooling".)

2. Uniqueness theorem. It is not hard to prove a general uniqueness theorem, involving boundary value problems with boundary conditions of Types I, II, or II′, which is adequate for most physical flow problems. To formulate it, let $u = u(A_k)$ and $i = i(a_i)$; and u' and i' represent two different solutions of the same boundary value problem. Consider the expression

$$\begin{aligned} D^* &= \sum_L (i_k - i'_k)(\Delta u_k - \Delta u'_k) \\ &= \sum_L [c_k(\Delta u_k) - c_k(\Delta u'_k)][\Delta u_k - \Delta u'_k]. \end{aligned} \tag{5}$$

In the linear case, clearly (5) simplifies to

$$D^* = \sum_L c_k(\Delta u_k - \Delta u'_k)^2. \tag{5′}$$

In any case, the following result is immediate:

LEMMA 1. If all the conductivity functions $c_k(\Delta u)$ satisfy (2a), then $D^* \geqq 0$. Strict inequality holds unless $u - u'$ is a constant.

We now make a second evaluation of D^*. By (3),

$$\begin{aligned} \sum_L [c_k(\Delta u_k) - c_k(\Delta u'_k)][\Delta u_k - \Delta u'_k] &= \sum_L \{[c_k(\Delta u_k) - c_k(\Delta u'_k)][\sum_N \epsilon_{hk}(u(A_h) - u'(A_h))]\} \\ &= \sum_N (\nu_h - \nu'_h)[u(A_h) - u'(A_h)]; \end{aligned}$$

where, for example,

$$\nu_h = \sum_L \epsilon_{hj} i(a_j)$$

is the "influx" corresponding to the potential u at the node A_h of N. It follows from (5) and (1) that

$$D^* = \sum_{\partial N} (\nu_h - \nu'_h)[u(A_h) - u'(A_h)]. \tag{5+}$$

For boundary value problems involving only conditions of Types I or II at the terminals, one has $D^* = 0$. If conditions of Types I, II, or II′ occur, provided that (4) is assumed, clearly $D^* \leqq 0$. Comparing with Lemma 1, we get

THEOREM 1. There is at most one solution to any boundary value problem defined by (1) and (2), with boundary conditions of Types I, II, or II′, provided that (2a)

[4]D'Auriac [14] and Duffin [17] consider the Dirichlet and Neumann problems, plus a special "mixed" problem, where a Dirichlet condition is imposed at some boundary nodes and a Neumann condition at the remainder of the terminals. D'Auriac proves uniqueness and Duffin, existence and uniqueness theorems.

and (4) are assumed and that, in the Neumann problem, potential functions which differ only by a constant are considered to be identical.

3. Dissipation function; variational principle for the Dirichlet problem. If i and u are any two functions defined on the oriented links and the nodes, respectively, of a network, we may define the *dissipation function* as the sum

$$D = \sum_L i(a_k)\Delta u(a_k). \tag{6}$$

(The name "dissipation function" expresses the fact that, in the two physical problems mentioned in Sec. 1, the expression D represents the rate of energy dissipation.) We shall now derive an alternative formula for D, analogous to (5+) for D^*. Since

$$\Delta u(a_k) = u(A_{i(k)}) - u(A_{f(k)}) = \sum_h \epsilon_{hk} u(A_h),$$

one has

$$\sum_L c_k(\Delta u_k)\Delta u_k = \sum_L \{c_k(\Delta u_k) \sum_N \epsilon_{hk} u(A_h)\} = \sum_N \nu_h u(A_h);$$

and thus

$$D = \sum_{\partial N} \nu_h u(A_h). \tag{6+}$$

The expression D, according to (6+), represents the rate of energy influx.

In the linear case, the dissipation function reduces to $\sum_L c_k(\Delta u_k)^2$, and it is classical[5] that this is minimized by the solution of the network problem over the class of potentials assuming the given terminal potentials. We shall now derive an analogous variational principle for the non-linear case. However, this will not, in general, involve the dissipation function.

To formulate the new variational principle, we suppose u is given on ∂N, but is unknown on $N - \partial N$. For any assumed values of u on $N - \partial N$, we can then satisfy (2) automatically by *defining* $i(a_k) = c_k(\Delta u_k)$ on each link a_k. It remains to satisfy (1), and for this we shall find a variational formulation. Namely, define the functions C_k by

$$C_k(\Delta u) = \int_0^{\Delta u} c_k(x)\, dx,$$

so that the derivatives

$$C_k'(\Delta u) = \frac{dC_k}{d(\Delta u)} = c_k(\Delta u);$$

for simplicity, we shall assume condition (2b). (Duffin [17, pp. 965-967] uses this same device of auxiliary functions for what we call the Neumann problem.)

THEOREM 2. For given $u(A_h)$ on ∂N (i.e. for the Dirichlet problem), assuming (2b), the first variation of

$$V(u) = \sum_L C_k(\Delta u_k) \tag{7}$$

is zero at each ("interior") node of $N - \partial N$ if and only if Kirchhoff's node law (1) holds at each interior node.

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[5]See W. Thomson [2]; J. C. Maxwell [3, vol. I, pp. 403-408].

Proof: (By the first variation is of course meant the following limit:

$$\delta V(u) = \frac{d}{d\epsilon} V(u + \epsilon\delta u) \Big|_{\epsilon=0},$$

where δu is any potential function defined on N but which vanishes on ∂N.) By direct computation, writing $a_k = A_{i(k)}A_{f(k)}$, one has

$$\begin{aligned} \delta V &= \sum_L C_k'(\Delta u_k)[\delta u(A_{i(k)}) - \delta u(A_{f(k)})] \\ &= \sum_L \{C_k'(\Delta u_k) \sum_N \epsilon_{hk}\delta u(A_h)\} \\ &= \sum_N \{\delta u(A_h) \sum_L \epsilon_{hk} i(a_k)\}. \end{aligned}$$

For A_h in ∂N, the number $\delta u(A_h)$ is zero, while for A_h in $N - \partial N$, the $\delta u(A_h)$ are arbitrary. The conclusion of Lemma 2 is now evident.

COROLLARY. If (2a), (2b) hold, then Kirchhoff's node law (1) holds if and only if $V(u)$, considered as a function of the arbitrary values of the potential at interior nodes, has an absolute minimum.

For if, regardless of (2a), the function $V(u)$ has even a local minimum, then $\delta V = 0$, and hence Kirchhoff's node law holds. While, on the other hand, if (2a) and (2b) hold, then $V(u)$ is a *convex* function of u, since it is a sum of convex functions either of the individual $u_h = u(A_h)$, or of pairs of these variables, as may be readily seen from (7). We leave the detailed verification of this to the reader. Hence, if Kirchhoff's law holds for some u, the convex function $V(u)$ must have an absolute minimum for this particular u.

Remark. In the case (2c) of an *exponential conductivity law*, with the same exponent α for all links in the network, the dissipation function D is proportional to the function V, and therefore D can be used in place of V in the results of this section.

4. Existence theory for the Dirichlet problem. We shall now derive an existence theorem for the Dirichlet problem which is adequate for most physical applications. In order to avoid giving the impression that it is the "best possible", we shall preface it by giving a much stronger result for the linear case.

In the linear case (2d), a given trial potential function, when used to construct i by means of $i(a_k) = c_k[u(A_{i(k)}) - u(A_{f(k)})]$, for each link $a_k = A_{i(k)}A_{f(k)}$, will satisfy Kirchhoff's node law (1); i.e., (see Sec. 3) will solve the Dirichlet problem, if and only if

$$\sum_k \epsilon_{hk}c_k[u(A_{i(k)}) - u(A_{f(k)})] = 0, \tag{8a}$$

for every A_h in $N - \partial N$. This gives a system of s linear equations in the s unknowns $u(A_h) = u_h$, which may be more compactly written thus

$$\sum_{i=1}^{n} c_{hi}u(A_i) = b_h, \qquad h = 1, \cdots, s, \tag{8}$$

where the numbers b_h are known. The matrix of coefficients $\| c_{hi} \|$ of (8), which is *symmetric* (as follows readily from (8a) and the definition of the incidence matrix $\| \epsilon_{hk} \|$) will be called the *conductivity matrix* of the network. It is well known[6] that for any system like (8), existence and uniqueness are equivalent to each other, and also to the

[6] G. Birkhoff and S. MacLane [25, chap. X].

condition that the determinant of the matrix of coefficients be different from zero. This gives the following result[7].

THEOREM. If (2d) holds, the Dirichlet problem is solvable, for a given network N (with at least one interior node and at least one boundary node) for arbitrary values, if and only if det $\| c_{hi} \| \neq 0$. This condition is also necessary and sufficient for uniqueness.

We now see how special, in the linear case, is the condition (2a) requiring all conductivities to be positive. If this condition holds, then all the diagonal elements c_{hh} are positive, while

$$c_{hh} \geqq \sum_{i \neq h} | c_{hi} |, \qquad \text{for} \qquad h = 1, \cdots, s,$$

with strict inequality holding if and only if the node A_h is linked directly to a boundary node (which will certainly occur for at least one node, since neither $N - \partial N$ nor ∂N is empty, and the network is connected). It follows[8] that, if, in addition the matrix $\| c_{hk} \|$ is not reducible to the form

$$\begin{Vmatrix} P & U \\ 0 & Q \end{Vmatrix}$$

by the same permutation of the order of the rows and columns, where the matrices P, U, Q, 0 are all square matrices, and 0 consists only of zeros, then det $\| c_{hk} \| \neq 0$. However, it is easy to see, again using the theorem mentioned in footnote 8, that if all the conductivities are positive, and the conductivity matrix of a *connected* network has the "exceptional" form just mentioned, then its determinant is still not zero. For if $\| c_{hk} \|$ is of this exceptional form then its determinant is the product of the determinants of P and Q, each of which is again symmetric and "dominantly diagonal", and may be further reduced, in the same way that the original conductivity matrix was reduced, in case either of them is exceptional. (Notice that, in view of the symmetry of $\| c_{hk} \|$, it follows that the submatrix U must consist only of zeros.) Continuing this reduction as far as possible until only non-exceptional symmetric matrices occur (only a finite number of steps are possible) one finds that the det $\| c_{hk} \|$ is the product of a finite number of determinants, each corresponding to a "dominantly diagonal" matrix which is not "exceptional", and that all elements not appearing in this product are zero. Since the given network is *connected*, at least one node in each subnetwork associated with these submatrices must be linked directly to a boundary node of the given network. Hence, by the theorem mentioned in footnote 8, the determinant of each subnetwork is not zero, and thus det $\| c_{hk} \|$ is not zero either. It is clear that this class of non-singular, dominantly diagonal symmetric matrices is but a very small subclass of the class of all non-singular symmetric matrices.

The existence theorem which we shall now prove for the (possibly) non-linear case corresponds, however, to the theorem (in the linear case) obtained from Theorem 2 upon making the superfluous additional assumption that the conductivity matrix $\| c_{hi} \|$ is "dominantly diagonal" in the sense just described above.

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[7]See C. Saltzer [27, p. 122], J. L. Synge [20, p. 127].

[8]See Theorem III of Olga Taussky [18, p. 673]. For an application to electrical networks, see M. Parodi [13].

By Theorem 2, *any* local minimum of $V(u)$ will provide a solution, in the non-linear or linear case. However, if every $C_k(\Delta u) \to +\infty$ as $|\Delta u| \to +\infty$, then $V(u)$ will be bounded below everywhere; and be arbitrarily large[9] outside any sufficiently large bounded "cube" in $(u_1, \cdots, u_s)$ space. Hence $V(u)$ will have an *absolute* minimum inside some such cube, by a theorem of Weierstrass on continuous functions. We conclude

THEOREM 3. If (2b) holds, and if, for all k,

$$\int_0^{\infty} c_k(x)\, dx = \int_0^{-\infty} c_k(x)\, dx = +\infty, \tag{9}$$

in the sense of improper Riemann integration, then the Dirichlet problem has a solution for arbitrary boundary values.

COROLLARY. If (2a) and (2b) both hold, then (9) may be replaced by the conditions

$$c_k(x) > 0, \qquad \text{for some} \qquad x > 0, \tag{9a}$$

and

$$c_k(x) < 0, \qquad \text{for some} \qquad x < 0. \tag{9b}$$

5. Neumann problem. The Neumann problem is *dual* to the Dirichlet problem, in the sense that the rôles of u and i are interchanged. To make the duality more marked, we note that, since any continuous, strictly increasing function $y = c_k(x)$ has a (unique) continuous, strictly increasing inverse function $x = r_k(y)$, conditions (2a), (2b) are self-dual. Accordingly, we shall replace (2) in Sec. 1 by

$$\Delta u_k = r_k[i(a_k)], \tag{10}$$

and refer to the r_k as *resistance* functions. The condition that there exists a single-valued potential $u(A_h)$, such that $\Delta u_k = u(A_i) - u(A_j)$ whenever $a_k = A_i A_j$, is evidently *Kirchhoff's circuit law*

$$\sum_{\Gamma} r_k[i(a_k)] = 0, \tag{11}$$

for any sequence Γ of oriented links forming a closed *cycle* (or circuit).

For a given influx ν on ∂N, satisfying the consistency conditions (3′), the most general current function i which satisfies Kirchhoff's circuit law (11) is obtained by "adding", onto some fixed current function satisfying the same conditions, "cyclic" currents $\beta_1, \cdots, \beta_t$ around closed cycles $\Gamma_1, \cdots, \Gamma_t$ forming a *basis* for the closed cycles of the network. This fact is easily seen in the case of a planar network (graph), when the basic cycles may be taken as the (oriented) boundaries of the polygons into which the network subdivides the plane, and one has $r + 1 = n + t$. The general case is also classic[10] (i.e., $t = r - n + 1$ for a connected graph).

Thus, once an initial current distribution satisfying (3′) and (11) has been found, each $\beta = (\beta_1, \cdots, \beta_t)$ determines a unique current distribution on the set of links L, satisfying (3′) and (11), while (10) may be taken as *defining* Δu. (We treat (10) as a substitute for (2), recalling that (2) and (10) are equivalent if (2a) and (2b) hold.)

We now define $R_k(i) = \int_0^i r_k(y)\, dy$, for each link a_k, so that the derivative $R_k'(i) = dR_k/di = r_k(i)$, and assume for simplicity that the $r_k(y)$ are continuous.

[9]This follows from a modification of an argument of Duffin [17, p. 965] which uses in an essential way the fact that the network is connected.

[10]Poincaré [4], Veblen [7, p. 9], Ingram and Cramlet [12, p. 137], and Synge [20, p. 123].

THEOREM 2′. For given consistent values [see (3′)] of ν_h on ∂N, the first variation of the function

$$W(\beta) = \sum_L R_k[i(a_k)] \tag{12}$$

vanishes identically if and only if the $r_k[i(a_k)]$ satisfy Kirchhoff's circuit law (11).

Proof: By direct computation,

$$\delta W = \sum_L R_k'[i(a_k)]\delta i(a_k) = \sum_B \delta\beta_i \sum_{\Gamma_i} r_k[i(a_k)], \tag{13}$$

where the last sum is taken over a basis B of the closed cycles of the network, which consists of $\Gamma_1, \cdots, \Gamma_t$. This last sum is zero for arbitrary $\delta\beta$ if and only if the individual sum taken over each Γ_i is zero, which is equivalent to (11).

COROLLARY. If (2a), (2b) hold, then Kirchhoff's circuit law (11) holds if and only if $W(\beta)$ has an absolute minimum.

The proof is similar to that of the corollary to Lemma 2, and may be omitted.

Remark. In the case (2c) of an exponential resistance law, with the same exponent α for all links in the network, it follows (see Sec. 3) that the dissipation function D is proportional to the function W, and therefore D can be used in place of W in the above results. This is known in the linear case[11].

THEOREM 3′. The Neumann problem has a solution for any set of compatible boundary influxes [see (3′)], provided that (2b) holds and that

$$\int_0^{\infty} r_k(y)\, dy = \int_0^{-\infty} r_k(y)\, dy = +\infty. \tag{14}$$

The proof is identical with that of Theorem 3, and the analogue of the corollary to Theorem 3 also follows similarly.

6. Mixed boundary value problem; relaxation methods; existence theorem. Without striving for maximum generality, we shall prove an existence theorem which is adequate for most applications. The method of proof to be employed is constructive, in that in many concrete instances the performance of the "relaxation steps" used in the proof can actually be used in order to construct numerically a solution to a network boundary value problem.

Let us suppose that our boundary conditions are of Types I and II′. Since Kirchhoff's node law (1) really corresponds to a condition of Type II′, with $F_h(u) \equiv 0$, we can reformulate our problem as that of satisfying a condition of Type II′ at *all* those nodes $A_1, \cdots, A_m$ where the potential $u(A_h)$ *is not* prescribed; we shall denote this set (supposed to be not empty) of nodes by M. Further, we shall suppose that the set of nodes at which the potential *is* prescribed, which is $N - M$, contains at least one node. It is for this class of boundary value problems that an existence theorem will be proved, under the assumption that (2a) and (2b) hold and that

$$\lim_{x \to -\infty} c_k(x) = -\infty; \qquad \lim_{x \to +\infty} c_k(x) = +\infty. \tag{15}$$

Thus we shall exclude the case of "saturation currents".

(The requirement that the set $N - M$ be non-empty, seemingly—but only seemingly—excludes the Neumann problem from consideration. Because, granting, for the purposes

[11] W. Thomson (Lord Kelvin) [2], and J. C. Maxwell [3].

of the present discussion, that an existence theorem has been proved for the above mentioned class of mixed problems, then an existence theorem for the Neumann problem readily follows from it. This can be seen by merely assigning arbitrarily the value of the potential at a fixed node of the network, as an additional boundary condition, besides the given Neumann conditions. By this obvious artifice, any Neumann problem can be turned into a mixed problem of the class described above, and hence has a solution for each arbitrarily assigned value of the potential at the chosen fixed node, i.e. a "one-parameter" family of solutions. In view of this, the Neumann problem need not be mentioned in the following discussion.)

Just as in Sec. 3, we can satisfy (2) by *fiat* for any choice of $u_1 = u(A_1)$, $\cdots$, $u_m = u(A_m)$, merely by defining $i(a_j) = c_j(\Delta u_j)$ for each link a_j. We can then compute $\nu_h = \sum_L \epsilon_{hj} i(a_j)$, for each node A_h in M, and define the *discrepancy* (or *residual*) function

$$\delta_h = \nu_h - F_h(u_h), \qquad h = 1, \cdots, m. \tag{16}$$

An existence theorem clearly asserts that $\delta(u) = 0$ for some $u = (u_1, \cdots, u_m)$.

LEMMA 4. The function $\delta = T(u)$ is one-to-one and continuous.

Proof: The continuity of T follows from the fact that the functions c_k are continuous by (2b), and the functions F_h are also continuous, by (4). It remains to show that distinct u determine distinct δ. This follows readily from Theorem 1, but we shall go over the proof, to emphasize the rôle of the requirement that the set $N - M$, where the potential values are assigned, is not empty. To this end, consider, as in (5) and (5+) that

$$D^* = \sum_L (i_k - i'_k)(\Delta u_k - \Delta u'_k) \geqq 0,$$

by (2a). On the other hand

$$\begin{aligned} D^* &= \sum_L \{(i_k - i'_k) \sum_N \epsilon_{hk}[u(A_h) - u'(A_h)]\} \\ &= \sum_N (\nu_h - \nu'_h)[u(A_h) - u'(A_h)], \\ &= \sum_M (\nu_h - \nu'_h)[u(A_h) - u'(A_h)] \end{aligned}$$

and if $T(u) = \delta = \delta' = T(u')$, then

$$D^* = \sum_M [F_h(u_h) - F'(u_h)][u(A_h) - u'(A_h)] \leqq 0,$$

by (4). Hence $D^* = 0$, and by Lemma 1, it follows that $u - u'$ is a constant. But this constant difference must be zero, since it is zero for each node in $N - M$, which is not empty.

Now, still assuming (2a), (2b) and (15), we pass on to a relaxation method. We shall consider the *residuals*[12] δ_h of a variable trial function $u(A_h)$, which are defined by (16). We first prove four lemmas involving the "order" relation.

LEMMA 5. As $u(A_h)$ is increased, all other values of u being held fixed, δ_h increases, all "adjacent" δ_k decrease, and all other δ_j remain constant.

Proof: For if A_k is "adjacent" to A_h, that is, there is either a link A_hA_k or a link A_kA_h in the network, then an increase in $u(A_h)$ increases [by (2a)] either $i(A_hA_k)$ or

[12]We shall conform to the terminology of R. V. Southwell [11], where possible.

$-i(A_kA_h)$, as the case may be; hence it increases ν_h, decreases ν_k, and leaves unchanged ν_j when A_j is not adjacent to A_h. Also, by (4), an increase in $u_h = u(A_h)$ either decreases or leaves unchanged $F_h(u_h)$, and leaves unchanged all remaining $F_j(u_j)$, where $A_j \neq A_h$.

LEMMA 6. Consider a node A_h, and suppose that the values of u at all adjacent nodes A_k are increased, while the values of u at A_h, and at all nodes not adjacent to A_h are held fixed. Then δ_h decreases, while δ_k, where A_k is adjacent to A_h, increases.

The proof follows along similar lines to that of Lemma 5.

Now, consider δ_h [see (16)] as a function of the single real variable $u(A_h)$, all the other $u(A_k)$ being kept constant. From (2b), (15), and (4) it follows that δ_h is a strictly increasing continuous function of $u(A_h)$, which varies continuously from $-\infty$ to $+\infty$ as $u(A_h)$ does the same. Hence there is exactly one choice of $u(A_h)$ which will make $\delta_h[u(A_h)] = 0$ ("liquidate the residual" at A_h), provided that all the other $u(A_k)$ are kept constant. We define (exact) *point relaxation* at each node A_h to consist of replacing the value $u(A_h)$ by this particular value which makes δ_h vanish at A_h, all the other values $u(A_k)$, for $A_k \neq A_h$, being kept constant.

LEMMA 7. Relaxation at a given node is isotone on trial solutions of the same problem (i.e. it preserves order).

Proof: The proof is by contradiction. Suppose that u and u' are such that $u(A_k) \geqq u'(A_k)$ for all A_k in N, and let $v_h = v(A_h)$, and $v'_h = v'(A_h)$ denote, respectively, the functional values obtained from u and u' by point relaxation at the node A_h. Suppose, contrary to what we wish to prove, that $v_h < v'_h$. Now, starting with the "relaxed" function u'_R (i.e. with the function whose value at each node $A_j \neq A_h$ is $u'(A_j)$, while at A_h its value is v'_h) one can proceed in two steps to the "relaxed" function u_R, and obtain a contradiction, as follows. First, replace the values of u' at all the nodes different from A_h by the corresponding values of u at these nodes, leaving the functional value unchanged at A_h itself, and denote the resulting "hybrid" function u'^*. By Lemma 6, and the definition of point relaxation, one has that

$$0 = \delta_h(u'_R) \geqq \delta_h(u'^*). \tag{17}$$

Secondly replace the value of u'^* at A_h, which is v'_h, by v_h, and leave the values of u'^* at all nodes different from A_h unchanged. The resulting function is precisely the "relaxed" function u_R. Since, by assumption, $v_h < v'_h$, it follows from Lemma 5 that

$$\delta_h(u'^*) > \delta_h(u_R). \tag{18}$$

But a comparison of inequalities (17) and (18) then shows that $\delta_h(u_R) < 0$, contradicting the fact that, since v_h was obtained by point relaxation of u at A_h, the number $\delta_h(u_R)$ must be zero. This completes the proof of Lemma 7.

In the proof of the following lemma and the theorem to follow we shall make use of two more additional assumptions, one concerning the conductivity functions c_k and the other concerning the functions F_h of (4). For convenience we write them as follows:

For every k, one has $c_k(0) = 0$, (19)

For every function F_h which does not vanish identically, there is a number x_1 such that $F(x_1) \leqq 0$ and a number x_2 such that $F(x_2) \geqq 0$. (20)

In view of (4), it follows from (20) that whenever F_h is not identically zero then it is $\geqq 0$ for all sufficiently negative x and that it is $\leqq 0$ for all sufficiently positive x. As for

(19), it certainly holds in the important special cases (2c) and (2d), and it means intuitively, that "if the potential is constant then there is no flow of current".

LEMMA 8. Suppose that (19) and (20) hold, in addition to (2a), (2b) and (15). Let u_0 be an arbitrary trial function (i.e. having the prescribed values on $N - M$). Then there exist two other trial functions v_0 and w_0 such that

$$v_0(A_h) \leqq u_0(A_h) \leqq w_0(A_h),$$

and

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m.$$

Proof: It will suffice to show how to construct the trial function v_0 such that both

$$v_0(A_h) \leqq u_0(A_h) \tag{21}$$

and

$$\delta_h(v_0) \leqq 0, \tag{22}$$

for $h = 1, \cdots, m$, since the construction of w_0 is entirely analogous. The function v_0 will be defined in the following manner:

$$v_0(A_h) = \begin{cases} C, & \text{for} \quad A_h \text{ in } M \\ u_0(A_h), & \text{for} \quad A_h \text{ in } N - M, \end{cases} \tag{23}$$

where C is a constant, which is to be chosen sufficiently negative so that the requirements (21) and (22) asked of v_0 are met. First of all, if $C \leqq \min_N u_0$ then (21) clearly holds. As for (22), notice that if A_h in M is *not* linked to any node of $N - M$, and $F_h \equiv 0$, then [by (19)] it follows from (23), for *any* choice of C in (23), that $\delta_h(v_0) = \nu_h(v_0) = 0$, and (22) holds; however, if $F_h \not\equiv 0$, then still $\nu_h(v_0) = 0$, so that [by (20)] by choosing C sufficiently negative it will be true that $\delta_h(v_0) = -F_h(v_0) \leqq 0$, and (22) will again hold. It remains to consider the case when A_h in M is linked to at least one node in $N - M$. From (3), in view of (19), it follows that, for *any* choice of C in (23), *only* the links joining A_h to a node of $N - M$ contribute essentially to the sum in $\nu_h(v_0)$, and from (2a), (2b), (4) it is seen that if $C = v_0(A_h)$ is sufficiently negative, then $\delta_h(v_0) = \nu_h(v_0) - F_h(v_0)$ will be $\leqq 0$, fulfilling (22). Thus, all in all, in order that v_0 defined by (23) fulfill the requirements (21), (22), one sees that C must satisfy a *finite* number of conditions, all of which may be made to hold simultaneously, if only C is chosen sufficiently negative. This completes the proof of Lemma 8.

We are now ready to prove our main result[13].

THEOREM 4. Suppose (2a), (2b), (15), (19), (20) hold. Let u_0 be any initial trial solution of a mixed network flow problem, and suppose $u_1, u_2, u_3, \cdots$ are obtained by successively "point-relaxing" the residuals of the initial trial solution u_0 at an infinite sequence of nodes of M, in such a way that each node A_h in M occurs infinitely often in the sequence of nodes. Then the sequence of trial functions $u_1, u_2, u_3, \cdots$ converges to the solution z of the given problem, the uniqueness of which has already been established in Theorem 1.

Proof: First, by Lemma 8, there exist trial functions v_0 and w_0 such that both

$$v_0(A_h) \leqq u_0(A_h) \leqq w_0(A_h),$$

and

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m.$$

[13]This generalizes directly a result of J. B. Diaz and R. C. Roberts [22].

Let v_1, u_1, w_1 denote the functions obtained from v_0, u_0, w_0, respectively, by point relaxation at the first node of the preassigned sequence of nodes. By Lemma 7, the initial sandwich order is preserved, i.e.

$$v_1(A_h) \leqq u_1(A_h) \leqq w_1(A_h), \qquad h = 1, \cdots, m.$$

As a matter of fact, since

$$\delta_h(v_0) \leqq 0 \leqq \delta_h(w_0), \qquad h = 1, \cdots, m,$$

it actually follows that (see Lemma 5)

$$v_0(A_h) \leqq v_1(A_h) \leqq u_1(A_h) \leqq w_1(A_h) \leqq w_0(A_h), \qquad h = 1, \cdots, m$$

and that

$$\delta_h(v_1) \leqq 0 \leqq \delta_h(w_1), \qquad h = 1, \cdots, m.$$

Similar inequalities hold for any positive integer n, if we denote by v_n, u_n, w_n, respectively, the functions arising from v_0, u_0, w_0, respectively, after *successive* point relaxation at the first n nodes of the preassigned sequence of nodes. Namely, we have

$$\begin{aligned} v_0(A_h) \leqq v_1(A_h) \leqq \cdots \leqq v_n(A_h) \leqq u_n(A_h) \leqq w_n(A_h) \leqq \cdots \\ \leqq w_1(A_h) \leqq w_0(A_h), \end{aligned} \tag{24}$$

and

$$\delta_h(v_n) \leqq 0 \leqq \delta_h(w_n), \qquad h = 1, \cdots, m.$$

Since, for each A_h in M, the sequence of numbers $v_0(A_h)$, $v_1(A_h)$, $\cdots$, $v_n(A_h)$, $\cdots$ is non-decreasing and bounded above [e.g., by $w_0(A_h)$] it follows that the following limit exists

$$v(A_h) = \lim_{n\to\infty} v_n(A_h), \qquad h = 1, \cdots, m. \tag{25}$$

For A_h on $N - M$ we have that $v(A_h)$ equals $u_0(A_h)$, which is exactly the value each function v_n has at A_h. Thus, to show that the function v is indeed a solution of the mixed problem, it only remains to show that

$$\delta_h(v) = 0, \qquad h = 1, \cdots, m. \tag{26}$$

To do this, consider A_h in M. Since A_h occurs infinitely often in the preassigned sequence of nodes employed in point relaxation, it follows that there is an infinite sequence of positive integers $n_1 < n_2 < n_3 \cdots$ such that

$$\delta_h(v_{n_k}) = 0, \qquad k = 1, 2, 3, \cdots.$$

But then from (25) and the continuity of δ (see Lemma 4), Eq. (26) follows.

By proceeding in a similar manner with the non-increasing, bounded below sequence of numbers $w_0(A_h)$, $w_1(A_h)$, $\cdots$, $w_n(A_h)$, $\cdots$ one obtains that the function w defined by

$$w(A_h) = \lim_{n\to\infty} w_n(A_h), \tag{27}$$

for A_h in N, is also a solution of the mixed problem. The uniqueness Theorem 1 then shows that $v = w$, *the* solution of the mixed problem and finally (24), (26), and (27) then show that u_n also converges to *the* solution of the mixed problem.

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—NOTES—

UNSTEADY VISCOUS FLOW IN THE VICINITY OF A STAGNATION POINT*

BY NICHOLAS ROTT (*Cornell University*)

Consider a steady two-dimensional "stagnation point" flow of a viscous incompressible fluid in the upper x, y-plane; let the flow be directed towards and limited by a plate in the plane $y = 0$, with the stagnation point at $x = y = 0$. The corresponding flow pattern is well known as an exact solution of the Navier-Stokes equations.

Now, in addition, let the plate perform a harmonic motion in its own plane, i.e., in the x direction, while the flow at $y \to \infty$ remains steady. It seems to have remained unnoticed that even in this case, the exact Navier-Stokes equations yield a soluble problem of the boundary-layer type. Using for the velocity components u and v in the x- and y-directions respectively,

$$u = axf'(\eta) + be^{i\omega t}g(\eta), \tag{1}$$

$$v = -(a\nu)^{1/2}f(\eta), \tag{2}$$

and for the pressure, p,

$$p = -\frac{\rho}{2}a^2x^2 - \rho\nu aF(\eta) + p_0\,, \tag{3}$$

where

$$\eta = y\left(\frac{a}{\nu}\right)^{1/2}. \tag{4}$$

(ν = kinematic viscosity), and introducing these expressions in the Navier-Stokes equations, the following set of equations is easily obtained:

$$f'^2 - ff'' = 1 + f''', \tag{5}$$

$$ikg + gf' - fg' = g'', \tag{6}$$

$$ff' = F' - f'', \tag{7}$$

where $k = \omega/a$ is a "reduced" frequency. Equation (6) and Eq. (7) result from the equation of motion in the x-direction, by putting the terms proportional to x and independent of x respectively equal to zero. It is seen that Eq. (5) for the steady part is independent of the superimposed "g-flow." With the boundary conditions, $f(0) = f'(0) = 0$ and $f'(\infty) = 1$, Eq. (5) has the well-known Hiemenz solution. The viscous pressure term F can also be computed independently of Eq. (6).

Since the function $f(\eta)$ is known, Eq. (6) can be solved for g with the boundary conditions $g(0) = 1$, $g(\infty) = 0$. Consider first the steady motion of the plate with constant velocity b, i.e., $\omega = k = 0$. The corresponding solution g_0, say, fulfills the equation,

$$g_0'' + fg_0' - f'g_0 = 0. \tag{8}$$

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The exact solution is

$$g_0 = \frac{f''(\eta)}{f''(0)} = .811f'' \tag{9}$$

or, the velocity profile is proportional to the shear distribution of the Hiemenz flow (see Fig. 1). Solution (9) obviously fulfills the boundary conditions since $f''(\infty) = 0$,

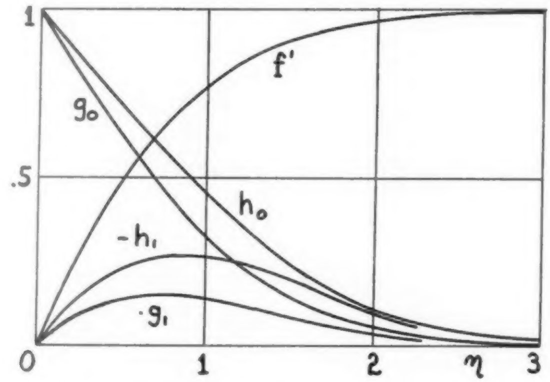

FIG. 1. Steady and quasi-steady velocity profiles.

and also the differential equation (8), because upon differentiating Eq. (5), the result is

$$f'''' + ff''' - f'f'' = 0.$$

With $f'''(0) = -1$, the shearing stress at the wall, τ_W, is proportional to

$$g_0'(0) = \frac{f'''(0)}{f''(0)} = -\frac{1}{f''(0)}, \tag{10}$$

so that the resulting τ_W from both the f- and the g-flow becomes

$$\tau_W = \rho(a\nu)^{1/2}\left\{axf''(0) - \frac{b}{f''(0)}\right\}. \tag{11}$$

Note that for $x = b/a$, the velocity outside the boundary layer is zero *relative to the wall*; the shearing stress at the wall, however, is zero at $x = .658b/a$. Evidently $\tau_W = 0$ does not necessarily mean separation for moving walls; in a system where the wall is at rest, the flow is not steady. The resulting boundary layer profiles are sketched in Fig. 2. Such development may be expected at the stagnation point of a Flettner rotor.

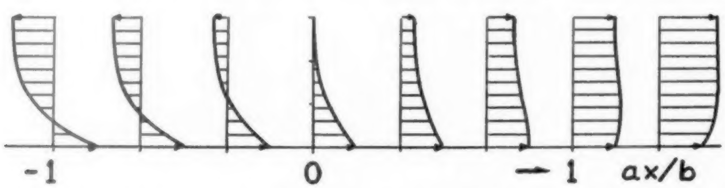

FIG. 2. Boundary layer development in the vicinity of a stagnation point with the wall moving at constant velocity b.

In the oscillating case, the two limiting cases, $k \ll 1$ and $k \gg 1$ will be considered. For $k \ll 1$, put

$$g = g_0 + ikg_1 + (ik)^2 g_2 + \cdots . \tag{12}$$

Equating powers of ik, the solution for g_0 is given by Eq. (9), obtained above; g_1 fulfills the equation

$$g_1'' + fg_1' - f'g_1 = g_0 = .811f'' \tag{13}$$

with the boundary conditions $g_1(0) = g_1(\infty) = 0$. One particular solution of the inhomogeneous equation (13) can be immediately given: $g_1 = .811f$. This function, however, does not fulfill the conditions at infinity, as $f(\infty) \to \eta$. The complete solution of Eq. (13) is obtained by adding the general homogeneous solution:

$$g_1 = \frac{f(\eta)}{f''(0)} + c_1 f'' \int_0^\eta \frac{E}{f''^2}\, d\eta + c_2 f'', \tag{14}$$

where

$$E = \exp\left(-\int_0^\eta f\, d\eta\right). \tag{15}$$

From $g_1(0) = 0$, it follows that $c_2 = 0$. The constant c_1 has to be adjusted so that $g_1(\infty) = 0$. Its value can be found with the help of the identity

$$f = f'' \int_0^\eta \frac{E}{f''^2}\, d\eta \int_0^\eta \frac{f''^2}{E}\, d\eta \tag{16}$$

which may be proved by identifying the proper inhomogeneous solution of Eq. (13) with f, or directly by using the properties of the function f as a solution of Eq. (5). It can be shown also that for $\eta \to \infty$, $f''^2/E \to 0$, and that the integrand of the inner integral in Eq. (16) is integrable between the limits 0 and ∞. Therefore, for large η, the inner integral in Eq. (16) will vary only slightly and may be replaced asymptotically by the constant value obtained after taking the limits between 0 and ∞:

$$f \to \left\{\int_0^\infty \frac{f''^2}{E}\, d\eta\right\} f'' \int_0^\eta \frac{E}{f''^2}\, d\eta. \tag{16a}$$

With this asymptotic expression, c_1 can be determined, and the final result is given by

$$g_1 = \frac{f}{f''(0)} - \left\{\frac{1}{f''(0)} \int_0^\infty \frac{f''^2}{E}\, d\eta\right\} f'' \int_0^\eta \frac{E}{f''^2}\, d\eta. \tag{17}$$

The curve $g_1(\eta)$ is also plotted in Fig. 1; its extremum is $-.152$. The shearing stress at the wall is proportional to

$$g_1'(0) = -\frac{1}{[f''(0)]^2} \int_0^\infty \frac{f''^2}{E}\, d\eta = -.496 \tag{18}$$

so that τ_W due to the g-flow is, to the first order in ik,

$$\tau_W = -\rho(a\nu)^{1/2} b(.811 + .496ik)e^{i\omega t}. \tag{19}$$

For high frequencies, $k \gg 1$, the WBK-method is appropriate. In Eq. (6), put

$$g = \exp\left(\int_0^\eta s\, d\eta\right) \tag{20}$$

then, s has to fulfill the equation

$$s' + s^2 + fs - f' = ik. \tag{21}$$

Now set

$$s = (ik)^{1/2}s_0 + s_1 + (ik)^{-1/2}s_2 + (ik)^{-1}s_3 + \cdots \tag{22}$$

which is also equal to

$$s = -\left(\frac{k}{2}\right)^{1/2} s_0 + s_1 - \left(\frac{1}{2k}\right)^{1/2} s_2 + \frac{1}{k}\left(\frac{1}{2k}\right)^{1/2} s_4 \cdots$$
$$- i\left\{\left(\frac{k}{2}\right)^{1/2} s_0 - \left(\frac{1}{2k}\right)^{1/2} s_2 + \frac{1}{k} s_3 - \frac{1}{k}\left(\frac{1}{2k}\right)^{1/2} s_4 \cdots\right\}. \tag{23}$$

Introduction of Eq. (22) into Eq. (21) gives upon equating powers of $(ik)^{1/2}$:

$$s_0 = 1, \qquad s_1 = -\tfrac{1}{2}f, \qquad s_2 = \tfrac{1}{8}f^2 + \tfrac{3}{4}f', \tag{24}$$
$$s_3 = -\tfrac{1}{8}ff' - \tfrac{3}{8}f'', \cdots .$$

The boundary condition $g(0) = 1$ is always fulfilled by the choice of limits in the integral in Eq. (20), and $g(\infty) = 0$ is assured by the selection of the proper sign if the double-valued quantity $(ik)^{1/2}$, namely, the one indicated in the decomposition Eq. (23). The resulting profile, using s_0 and s_1 only, is

$$g = \exp\left[-\left(\frac{k}{2}\right)^{1/2} i\eta\right] \exp\left[-\left(\frac{k}{2}\right)^{1/2} \eta\right] \exp\left(-\frac{1}{2}\int_0^\eta f\, d\eta\right). \tag{25}$$

The first two factors represent the "Stokes-solution" [1], which would be obtained by complete disregard of the f-flow. The last factor has the value of about .92 for $\eta = 1$ and .59 for $\eta = 2$. If, for large values of k, the second factor in Eq. (25) already assures a strong decay with η, the last factor remains very close to 1 in the whole domain where significant values of g may be found.

The shearing stress at the wall is

$$\tau_W = \rho(a\nu)^{1/2} b s(0) e^{i\omega t}. \tag{26}$$

Now the value of $s(0)$ differs from the Stokes value only if terms up to s_3 are included. From Eqs. (24), Eq. (23) gives

$$s(0) = -\left(\frac{k}{2}\right)^{1/2} - i\left[\left(\frac{k}{2}\right)^{1/2} - \frac{3}{8k} f''(0)\right]. \tag{27}$$

Therefore, the ratio of the out-of-phase shearing stress, τ_{W_i}, to the in-phase component, τ_{W_r}, is

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1 - .654k^{-3/2}, \tag{28}$$

whereas for low values of k, from Eq. (19), the ratio is

$$\frac{\tau_{W_i}}{\tau_{W_r}} = .612k. \tag{29}$$

In Fig. 3, the limiting cases for large and small k, Eqs. (28) and (29), are plotted, together with an estimated curve which joins the two plots smoothly. It is seen that for

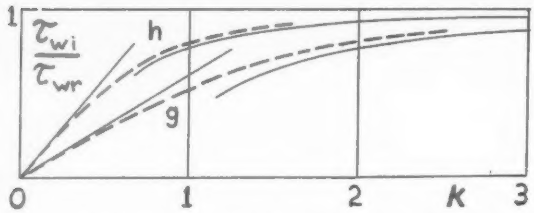

FIG. 3. The ratio of the out-of-phase and the in-phase components of the shearing stress at the wall as function of the reduced frequency k, for oscillations in x-direction (g) and z-direction (h).

a reduced frequency k larger than 2, the Stokes' value $\tau_{W_i}/\tau_{W_r} = 1$ gives errors less than 20%.

For the case of complex ω, or exponential acceleration, only one special case will be considered, for which the solution may be obtained without any computational effort. Put $k = -i$, i.e., let the time-dependence of the plate velocity be given by the factor e^{at}. The corresponding special equation for g is

$$g'' + fg' - (f' + 1)g = 0 \tag{6a}$$

which has the solution

$$g = 1 - f' \tag{30}$$

in view of Eq. (5). The complete solution is

$$u = axf'(\eta) + be^{at}[1 - f'(\eta)]$$

and the wall shearing stress is

$$\tau_W = \rho(a\nu)^{1/2}f''(0)[ax - be^{at}]. \tag{31}$$

Here it is seen that, in contrast to the case $k = 0$, Eq. (11), the zero of the shearing stress and the zero of the relative velocity between plate and fluid occur at the same value of $x[=(b/a)e^{at}]$.

Now let the plate move in the z-direction perpendicular to the xy-plane, i.e., the plane of flow. If the plate motion is uniform, the case under consideration is identical with the problem of the stagnation point in yawed flow, which has been solved by Prandtl [2] and Sears [3]. For unsteady motion of the plate in z-direction, Wuest [4] has already pointed out that the problem is soluble, and has given some numerical examples. Both cases become exact solutions of the Navier-Stokes equations, if the basic steady flow is the stagnation-point flow in the half-plane, as is presently assumed. If w, the velocity component of the flow in the z-direction, is taken in the form

$$w = ce^{i\omega t}h(\eta), \tag{32}$$

then Eqs. (5) to (7) remain unaffected, and the Navier-Stokes equation in z-direction yields

$$ikh - fh' = h'' \tag{33}$$

with the boundary conditions $h(0) = 1$, $h(\infty) = 0$. For $\omega = k = 0$, Prandtl's solution is obtained:

$$h_0 = \frac{\int_\eta^\infty E\, d\eta}{\int_0^\infty E\, d\eta} \tag{34}$$

[see Eq. (15)]. A series development for small k analogous to Eq. (12) leads to the equations

$$h_0'' + fh_0' = 0, \qquad h_1'' + fh_1' = h_0\,, \qquad h_2'' + fh_2' = h_1\,, \cdots . \tag{35}$$

The solution of the second equation, adjusted to the boundary conditions $h_1(0) = h_1(\infty) = 0$, is:

$$h_1 = h_0 \int_0^\eta \frac{(1 - h_0)h_0}{h_0'}\, d\eta + (1 - h_0) \int_\eta^\infty \frac{h_0^2}{h_0'}\, d\eta. \tag{36}$$

The profiles h_0 and h_1 are plotted in Fig. 1. The shearing stress at the wall in the z-direction becomes, to a first approximation in k:

$$\tau_W = -\rho(a\nu)^{1/2}c(.571 + .685ik)e^{i\omega t}. \tag{37}$$

The investigation of the flow-behavior for $k \gg 1$ follows completely the method already employed for the g-flow. Putting

$$h = \exp\left(\int_0^\eta r\, d\eta\right). \tag{38}$$

in Eq. (33) yields

$$r' + r^2 + fr = ik \tag{39}$$

and the series development analogous to Eq. (22) gives

$$\begin{gathered} r_0 = 1, \qquad r_1 = -\tfrac{1}{2}f, \qquad r_2 = \tfrac{1}{8}f^2 + \tfrac{1}{4}f', \\ r_3 = -\tfrac{1}{8}ff' - \tfrac{1}{8}f'', \cdots . \end{gathered} \tag{40}$$

It is seen that $r_0 = s_0$, $r_1 = s_1$.

The shearing stress at the wall again differs from the Stokes value only if r_3 is included; to this approximation,

$$r(0) = -\left(\frac{k}{2}\right)^{1/2} - i\left[\left(\frac{k}{2}\right)^{1/2} - \frac{1}{8k}f''(0)\right] \tag*{41)}$$

and we obtain

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1 - .215k^{-3/2} \tag{42}$$

whereas for $k \ll 1$, from Eq. (37),

$$\frac{\tau_{W_i}}{\tau_{W_r}} = 1.20k. \tag{43}$$

As before, both limiting cases are plotted in Fig. 3, together with an estimated smooth transition curve. It is seen that the Stokes limit is approached faster by the h-flow than by the g-flow; for $k = 1$ the deviation for the h-flow is about 20%.

A further set of exact solutions may be obtained if the basic steady, two-dimensional,

stagnation-point flow is replaced by a three-dimensional one. It is not necessary to assume rotational symmetry; the "potential" flow along the plate may be of the form $u = a_1 x$, $w = a_2 z$ with $a_1 \neq a_2$. The steady viscous solution has been obtained by Howarth [5]. If the plate moves or oscillates in any direction in the xz-plane, further exact solutions of the Navier-Stokes equation are easily obtained. No examples will be carried out for the three-dimensional stagnation point, as the cases treated before are well representative of the phenomena that may be expected.

It is interesting to note that the heat transfer to the plate remains unaffected by the motion of the plate in its plane, if the plate temperature is constant. The temperature field is obtained as a solution of the equation (in two dimensions)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{44}$$

where T is the temperature and σ is the Prandtl number; dissipation is neglected, as is permissible for high temperature differences between the fluid and the plate. For constant plate temperature (and constant temperature difference between the plate and the fluid), the solution with the property $\partial T/\partial x = 0$ is appropriate. But the plate motion under consideration only affects u and leaves v unchanged, so that for $\partial T/\partial x = 0$ no effect on the solution of Eq. (44) will be felt.

If unsteady heat transfer is enforced by a variable temperature on the plate, which becomes a function of time but not of x, the resultant equation becomes completely analogous to that of the h-flow, discussed before. For $\sigma = 1$, the same solution can be used as before; modifications for $\sigma \neq 1$ are easily obtained.

A case which has not been considered so far in this paper is the problem associated with a plate oscillating perpendicular to its plane, or the equivalent case of the oscillating basic stagnation-point flow. If the f-flow is unsteady, a $\partial f'/\partial t$ — term appears in the non-linear equation (5), so that solutions with a harmonic time-dependence can be obtained only to a linearized approximation. Ultimately, however, the unsteady part of the flow will influence the development of the steady part of the f-flow, due to the non-linearity of Eq. (5).

The linearized approximation, or the superposition of a small oscillating part to a basically steady f-flow has been investigated by Lighthill [6] as a special case of fluctuating flow problems with arbitrary velocity distributions. The linearized solution exhibits essentially the same features as found in the cases discussed before, inasmuch that a quasi-steady type for low frequencies and a high-frequency type approaching Stokes' solution can be distinguished. Lighthill finds as a "limit" between these cases, the value of $k = 5.6$ for the reduced frequency, i.e., higher than the limits which have been found above for the g- and h-flows. Lighthill also discussed the effects of the time-dependent f-flow on heat transfer, which does not vanish, as the v-component of the flow is affected.

Considering the flow at the stagnation-point of an oscillating airfoil with nose-radius R, the value of a is about U/R. For all reduced frequencies of practical interest in airfoil flutter, the high frequency approximation is appropriate for all unsteady boundary layer phenomena.

It may be noted that unsteady rigid rotary motion of the plate in its own plane leads to problems related to the well-known Kármán-Cochran case and its generalizations; again, however, solutions with harmonic time-dependence can be found only to a linearized approximation.

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STABILITY OF SPHERICAL BUBBLES*

By GARRETT BIRKHOFF (*Harvard University*)

The perturbation equations for a spherical bubble of radius $b(t)$ are [1, p. 306]

$$b(t)b_h'' + 3b'(t)b_h' - (h - 1)b''(t)b_h = 0. \tag{1}$$

It is the purpose of this note to give a general stability criterion for the stability of (1). Rewriting (1) in the form

$$x'' + p(t)x' + q(t)x = 0, \qquad p = 3b'/b, \qquad q = -(h - 1)b''/b, \tag{2}$$

we consider first the formal identity

$$\frac{d}{dt}\{x^2 + qx'^2\} = -\frac{x'^2}{q^2}[2pq + q'], \tag{3}$$

which is an easy consequence of (2).

Theorem 1. If $q < 0$, or if $q > 0$ and $2pq + q' < 0$, then (2) is *unstable*. If $q > 0$ and $2pq + q' > 0$, then (2) is *stable*.

Proof. If $q(t) < 0$, and $x(t)$, $x'(t)$ have the same sign for $t = t_0$, then they have the same sign for all $t > t_0$. This is evident since $x'(t_1) = 0$ offers the only possibility for the first sign change, and it implies $x''(t_1) = -qx(t_1)$, whence $x'(t_1 + dt) = -q(t_1)x(t_1)dt$ has the same sign as $x(t_1 + dt)$. Hence $x(t)$ grows forever in magnitude; this is of course the *non-oscillatory case.*

If $q(t) > 0$, then we are in the *oscillatory* case. To see this, replace (2) by the self-adjoint

$$d(Px')/dt + Qx = 0, \qquad P = \exp\left(\int p\, dt\right), \qquad Q = q \exp\left(\int p\, dt\right). \tag{4}$$

Then we use the Bocher-Prüfer variable Θ, defined by $\tan \Theta = -Px'/x$. Differentiating $\tan \Theta$, using (4), and simplifying, we get

$$d\Theta/dt = Q(t) \cos^2 \Theta + \frac{1}{P(t)} \sin^2 \Theta > 0. \tag{5}$$

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Hence (if $P(t)$ is bounded, and $Q(t)$ bounded away from zero), x' will assume the value 0 infinitely often, whenever $\Theta = \pi, 2\pi, 3\pi, \cdots$. Now, considering (3), we see that the amplitude of successive oscillations increases or decreases as stated in Theorem 1.

Remarks. Theorem 1 does not seem to be in the literature[1], which is mostly concerned with the special case $p = 0$. It does not cover cases, like that of the Mathieu equation $x'' + (a + b \cos t)x = 0$, in which the sign of $(2p + q')$ oscillates rapidly. In the special case that q is constant, then the conditions of Theorem 1 reduce to the familiar conditions $p > 0$ (positive damping) and $q > 0$ (positive restoring force).

Finally, in the neighborhood of a *regular singular point*, where $tp(t) = a_1 + \cdots$ and $t^2q(t) = a_2 + \cdots$ expanded in MacLaurin series, the condition $q > 0$ reduces in the limit to $a_2 > 0$, while $2pq + q' > 0$ reduces to $2a_2(a_1 - 1)/t^3 + \cdots > 0$, or (in view of $a_2 > 0$ and $t < 0$) to $a_1 < 1$. Thus we recapture the stability conditions $a_1 < 1$ and $a_2 > 0$ proved previously [1, p. 308] by use of the indicial equation. In summary, the stability conditions of Theorem 1 are a direct generalization of the classical Routh-Hurwitz conditions.

Now, substituting into Theorem 1 from the case (1)-(2) of a spherical cavity, with negligible surface tension and cavity density, we get the stability criteria $-b''/b > 0$ and $(-6b'b'' - bb''' + b'b'')/b^2 > 0$. Since $b > 0$, these are equivalent to

$$b'' < 0, \qquad (bb''' + 5b'b'') < 0; \tag{6}$$

the second inequality is equivalent to requiring that (b^5b'') be a decreasing function.

Evidently, the first inequality is Taylor's original criterion, that acceleration of the interface be towards the lighter fluid. The second inequality expresses the condition that there be *positive damping*[2].

Finally, we consider the general case treated recently by Plesset[3]. In Plesset's notation, $b = R$ and the basic equation involved can be written

$$b_h'' + (3R'/R)b_h' - Ab_h = 0. \tag{7}$$

Substituting in Theorem 1, we get the two stability conditions

$$A < 0 \quad \text{and} \quad 6AR' + A'R < 0, \tag{8}$$

of which the second is equivalent to the condition that R^6A be a decreasing function.

Historical remarks. The stability of spherical bubbles was apparently first considered by Riabouchinsky (Proc. Int. Congr. Appl. Mech., Stockholm (1930), p. 149). Riabouchinsky suggested that the spherical shape was stable when the bubble pressure was less than the ambient pressure, and unstable when it exceeded the ambient pressure. Later [Proc. Roy. Soc. A201 192, (1950)], Sir Geoffrey Taylor proposed the more accurate

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[1]See R. Bellman, *Stability theory of differential equations*, New York, 1953, and references given there. However, C. T. Taam (Proc. Am. Math. Soc. **5**, 705-15, Thm. 2 (1954)) has recently obtained a closely related result.

[2]The fact that negative damping is an important destabilizing factor in cavity collapse was pointed out by the author in July, 1952, at the Underwater Ballistics Conference in Pennsylvania State College.

[3]M. S. Plesset, J. Appl. Phys. **25**, 96-8, formula (13) (1954). Plesset's assertion that his $G(t) < 0$ implies stability, seems to be unjustified. A much more elaborate study of bubble instability has recently been given by R. H. Pennington, Tech. Res. Rept. 22 (1954), Appl. Math. and Statistics Lab., Stanford University.

condition that there was stability when $b'' < 0$, and instability when $b'' > 0$. The need for a second condition was suggested by the author[2] in 1952; in [1], the case of a vapor-filled bubble was treated. The present note supplies such a condition for the general case that b changes by a large ratio.

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ERROR BOUNDS FOR A NUMERICAL SOLUTION OF A RECURRING LINEAR SYSTEM*

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1. Introduction. Suppose that we are given a bounded region R, $f \geq 0$, a function ϕ_B on the boundary B of R, and point functions f and g in R, and that we are required to determine ϕ so that

$$(P): \quad \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = f\phi + g \text{ in } R, \qquad \phi = \phi_B \text{ on } B.$$

In order to calculate ϕ approximately, we first construct a square grid in a rectangle S that has sides parallel to the coordinate axis and is large enough to contain R in its interior. Then we approximate (P) by a system of linear algebraic difference equations, arriving at a nonhomogeneous linear algebraic system

$$A\phi = \eta, \tag{1}$$

where ϕ is a vector having a component associated with each of the N grid points inside R, η is a known N-vector, and A is a known $N \times N$ matrix. Finally, we obtain an approximate solution ϕ_0 of (1), committing an error

$$\epsilon = A^{-1}(\eta - A\phi_0).$$

A direct estimate of ϵ is given in Eq. (4) below. Our principal object is to show that ϵ can be estimated far more conveniently, though less exactly, from a properly chosen system L of linear algebraic equations set up on the $M \geq N$ grid points interior to the rectangle S. Though motivated and illustrated by problem (P), L can be constructed as soon as A (with the properties listed below) is known. This method applies generally to any system (1) with those properties.

2. Estimate for ϵ. We first list properties of $A = (a_{ij})$:

$$a_{ii} < 0, \quad i = 1(1)N; \qquad a_{ij} \geq 0, \quad i \neq j, \quad i, j = 1(1)N;$$

$$\sum_{j=1}^{N} a_{ij} \leq 0, \qquad i = 1(1)N,$$

the inequality holding for at least one value of i;

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the matrix A cannot be transformed by the same permutation of rows and columns to the form

$$\begin{pmatrix} P & U \\ 0 & Q \end{pmatrix},$$

where P and Q are square matrices, and 0 consists of zeros. It follows from Theorem II in [1] that $|A| \neq 0$.

We now develop a series expression for ϵ. Let m be the diagonal matrix with diagonal entries $m_i = -a_{ii}$, $i = 1(1)N$. It may be shown by induction that

$$A^{-1} = -\sum_{p=0}^{n-1} D^p m^{-1} + D^n A^{-1}, \tag{2}$$

where $D = I + m^{-1}A$. Let λ_q, $q = 1(1)N$, be the characteristic roots of D. From Theorem II in [1], it is seen that for $|\lambda| \geq 1$, the determinant $|D - \lambda I| \neq 0$; therefore, $|\lambda_q| < 1$ for $q = 1(1)N$. This implies that $D^n \to 0$ as $n \to \infty$. Hence, if ϕ_0 is an approximate solution to (1), the error in ϕ_0 being $\epsilon = (e_i)$, we have from (1) and the limit in (2) that

$$\epsilon = -\sum_{p=0}^{\infty} D^p m^{-1} \rho, \tag{3}$$

where the residual vector $\rho = (r_i)$ is $\eta - A\phi_0$.

We proceed to give a direct estimate for ϵ. Use is made of the abmatrix; see [2]. Let $B = (b_{pq})$ be a matrix. The matrix $\alpha(B) = (|b_{pq}|)$ is called the abmatrix of B. If $B = (b_{pq})$ and $C = (c_{pq})$, $p = 1(1)M$, $q = 1(1)N$, $\alpha(B) \geq \alpha(C)$ means $|b_{pq}| \geq |c_{pq}|$ for all p, q. We see from these definitions that $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ and $\alpha(AB) \leq \alpha(A)\alpha(B)$, provided the indicated operations are permissible. Finally, when $b_{pq} \geq 0$ for all p, q, $B = \alpha(B)$; and the prefix α will not be used. With this notation, since $D \geq 0$, $m_i > 0$, $m_i^{-1} \leq m_L^{-1}$, and $\alpha(\rho) \leq \psi_N r_U$, where $m_L = \min(m_i)$, $r_U = \max |r_i|$, and $\psi_N = (1, 1, \cdots 1)'$ in N dimensions, we have from (3) that

$$\alpha(\epsilon) \leq (r_U/m_L) \sum_{p=0}^{\infty} D^p \psi_N. \tag{4}$$

Let us now consider an $M \times M$ matrix $E \geq 0$, $M \geq N$, which contains an $N \times N$ submatrix $E_R \geq D$ and which satisfies $E^p \to 0$ as $p \to \infty$. Let $\psi_M = (1, 1 \cdots)'$ in M dimensions. We see that

$$\sum_{p=0}^{\infty} (E^p \psi_M)^* \geq \sum_{p=0}^{\infty} D^p \psi_N, \tag{5}$$

where $(E^p\psi_M)^*$ is the column vector formed by the N elements of $E^p\psi_M$ corresponding to the N rows of E which contain the submatrix E_R. Since $E^p \to 0$ as $p \to \infty$, $(I - E)$ is non-singular. We may verify that

$$\sum_{p=0}^{\infty} E^p = (I - E)^{-1}. \tag{6}$$

Hence,

$$\tau = \sum_{p=0}^{\infty} E^p \psi_M \tag{7}$$

is the solution of the system

$$(I - E)\tau = \psi_M . \tag{8}$$

Therefore, if $\tau^* = \sum_{p=0}^{\infty} (E^p\psi_M)^*$, we see from (4), (5), and (6) that

$$\alpha(\epsilon) \leq \tau^* r_U/m_L , \tag{9}$$

which is an estimate of the error ϵ in the approximate solution ϕ_0 of (1). In applying this bound, we choose E in such a way that (8) has a readily obtainable solution.

3. Application to a Poisson-type equation. To illustrate use of the methods in part 2, we return to the problem (P) in part 1 and its approximation by finite differences. Let the grid spacing in S be $\Delta x = \Delta y = \delta$. Assign one of the numbers $q = 1(1)N$ as an index to each of the grid points interior to R. The first order system approximating (P) then is:

$$-m_q\phi_q + T_q(\phi) = g_q\delta^2 - L_q(B), \qquad q = 1(1)N, \tag{10}$$

where:

$$m_q \geq 4;$$

$T_q(\phi)$ is the sum of values of ϕ on non-boundary points adjacent to the qth point; $L_q(B)$ is a linear form of boundary values required only where the qth point is adjacent to a boundary. An example will make this situation clear. With first order linear approximation throughout, the equation associated with ϕ_2 in Figure 1 is

$$\phi_1 - [4 + f_2\delta^2 + c/(1 - c)]\phi_2 + \phi_3 + \phi_4 = g_2\delta^2 - \phi_B/(1 - c).$$

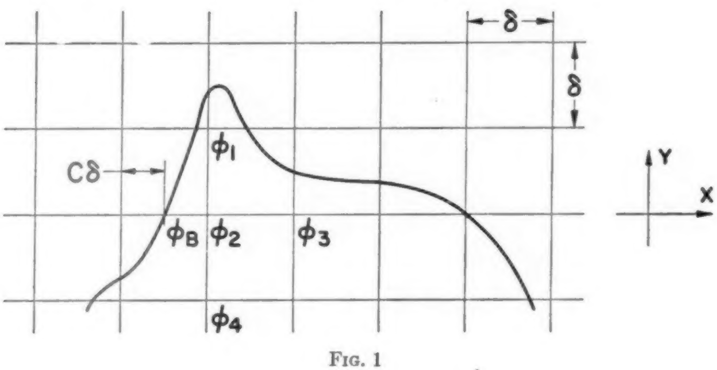

FIG. 1

We see that the qth row of the matrix D associated with (10) has elements m_q^{-1} corresponding to the unit coefficients of $T_q(\phi)$, all other elements of D being zero. Let the sides of the rectangle S be of length $H\delta$ and $I\delta$. Consider the system (11) corresponding to the $(H - 1) \times (I - 1)$ grid points interior to S:

$$\begin{gathered} t_{h,i} - m_L^{-1}(t_{h,i+1} + t_{h+1,i} + t_{h,i-1} + t_{h-1,i}) = 1, \qquad i = 0(1)I, \qquad h = 0(1)H, \\ t_{h,0} = t_{h,I} = t_{0,i} = t_{H,i} = 0, \end{gathered} \tag{11}$$

where $m_L = \min(m_q)$. Write the system (11) in the matrix form (8), the equations corresponding to the grid points interior to R being written in the same order as in (10). We see that the resulting matrix E then contains an $N \times N$ submatrix $E_R \geq 0$ with

entries m_L^{-1} in the same positions as the entries m_q^{-1} of the matrix D associated with (10). Hence, $E_R \geq D$. From Theorem II in [1], we see that the characteristic roots of E are less than one in absolute value; hence, $E^p \to 0$ as $p \to \infty$. It follows that the solution of (11) may be used in (9) to compute an error bound for an approximate solution of (1). We may verify that the solution of (11) is

$$t_{h,i} = (HI)^{-1} \sum_{r=1,s=1}^{r=H-1,s=I-1} C_{rs} \sin hr\pi/H \sin is\pi/I, \tag{12}$$

where

$$C_{rs} = \{[1 - (-1)^r][1 - (-1)^s] \sin r\pi/H \sin s\pi/I\} \cdot \{(1 - \cos r\pi/H)(1 - \cos s\pi/I)[1 - 2m_L^{-1}(\cos r\pi/H + \cos s\pi/I)]\}^{-1}.$$

For specifying the largest absolute residual which may be tolerated in an iterated solution, a bound for the maximum error $e_U = \max | e_i |$ is useful. From (9),

$$e_U \leq t_U r_U / m_L , \tag{13}$$

where $t_U = \max(t_{h,i})$. For simplicity, let both H and I be even. We may verify that t_U occurs at $h = H/2$, $i = I/2$, and that

$$t_U = -4(HI)^{-1} \sum_{r=1,s=1,\text{odd}}^{r=H-1,s=I-1} (-1)^{(r+s)/2}[1 - 2m_L^{-1}(\cos r\pi/H + \cos s\pi/I)]^{-1} \cdot \cot r\pi/2H \cot s\pi/2I, \tag{14}$$

where the summation is on odd values of r and s. For illustration, t_U in (14) with $m_L = 4$ is tabulated in Table 1 for $H, I = 8$ (2) 20. Since t_U in (14) increases as m_L decreases, the tabulated values may be used in (13) to state error bounds for an approximate solution of the linear system (10).

TABLE 1

Factor t_U for use in (13)

$H, I = 8(2)\ 20$

H/I	8	10	12	14	16	18	20
8	18.6						
10	22.7	29.2					
12	25.6	34.5	42.2				
14	27.6	38.5	48.6	57.5			
16	29.0	41.5	53.8	65.1	75.2		
18	30.0	43.8	57.9	71.4	84.0	95.2	
20	30.6	45.4	61.1	76.6	91.4	105	118

Acknowledgements. Mr. E. B. Carter of the Numerical Analysis Department at K-25 computed Table 1. The referee's comments simplified and generalized the original results.

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SOUND TRANSMISSION FROM A TUBE WITH FLOW*

By G. F. CARRIER (*Harvard University*)

1. Introduction. The increasing technological importance of thermo-acoustic phenomena in combustion chambers and other apparatus leads naturally to the need for understanding the transmission and reflection of acoustic waves at the inlet and exit sections of tubes through which a gas is flowing at moderate Mach number. In this paper, we treat this question for the inviscid perfect gas. The flow fields adopted are the simplest which are reasonable approximations to interesting physical situations and which lead to tractable problems. The problems are resolved by an extension of the work of Schwinger and Levine [1] (their work is essentially the $M = 0$ problem), and, in some interesting cases, our results can be expressed directly in terms of theirs. The modifications in these cases are reminiscent of subsonic aerodynamic results obtained using the Prandtl-Glauert method.

2. Formulation of the problems. The configuration to be treated here is depicted in Fig. (2.1). The steady flow through and about the tube is characterized by the specifi-

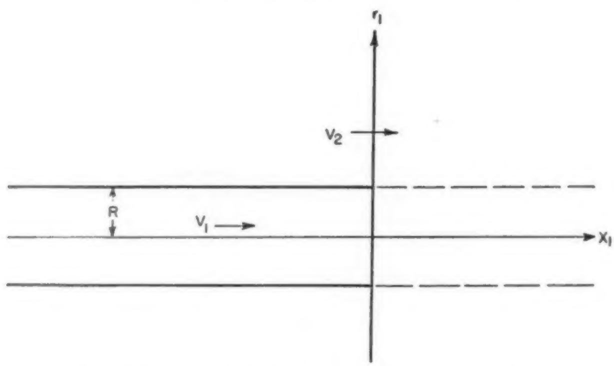

FIG. 2.1. Geometry of sound transmission problem.

cation of the thermodynamic state variables and the velocity field; the velocity field, in turn, is characterized by a single non-vanishing velocity component in the x_1 direction. This velocity has the value v_1 for $r_1 < R$ and v_2 for $r_1 > R$. In two of the problems of interest $v_1 = v_2$ and the state variables are the same in each region. In the third $v_1 > v_2$ and the state variables may differ in the two domains. The first two cases are distinguished according to whether v_1 is greater or less than zero.

If we restrict our analysis to small disturbances, assuming a perfect, inviscid, non-heat conducting gas, the equations governing the propagation of the disturbance in each region take the well known form

$$\rho \operatorname{div} \mathbf{v}' + \rho'_t + v\rho'_{x_1} = 0, \tag{2.1}$$

$$\rho(\mathbf{v}'_t + v\mathbf{v}'_x) + \operatorname{grad} p' = 0. \tag{2.2}$$

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Here, the primed quantities ($\mathbf{v}'$, ρ', p') are the perturbations in velocity, density and pressure, ρ is the steady state (constant) density, and the isentropic pressure-density law implies that $p' = a^2\rho'$ where a is the acoustic speed associated with the steady flow state; x_1, r_1 are the space coordinates. Each of these quantities will require the subscript 1 or 2 to distinguish the domain to which it applies when we discuss the flow in which $v_1 \neq v_2$.

It is convenient to introduce the following notation: $\mathbf{v}' = aR' \operatorname{grad} \phi$, $R' = R(1 - M^2)^{1/2}$, $s = x_1/R'$, $y = r_1/R$, $k^2 = \omega^2R^2/a^2(1 - M^2)$, and $\phi = \psi(s, y) \exp\{-i\omega[t + Mx_1/a(1 - M^2)]\}$. It follows from these and the foregoing that

$$\Delta\psi + k^2\psi = 0 \tag{2.3}$$

and

$$p' = -\rho a^2(M\psi_s - ik\psi) \exp\{-i\omega[t + Mx_1/a(1 - M^2)]\}, \tag{2.4}$$

where Δ is the cylindrical coordinate Laplace operator.[1] The boundary conditions we shall use differ in the three problems. If problem I is characterized by $v_1 = v_2 > 0$, the essential information is: $p'(x, R, t)$ is continuous on $s > 0$, $\psi_y(s, 1) = 0$ on $s < 0$. Note that, as in oscillating airfoil theory, one can anticipate that $\psi(s, 1)$ will not be continuous for $s > 0$. In problem II, $v_1 = v_2 < 0$ and both p' and ψ are continuous on $y = 1$, $s > 0$; again, $\psi_y(s, 1) = 0$ when $s < 0$. For problem III, with v_1 and v_2 positive but not equal, the boundary conditions are the same as those of problem I.

3. The continuous flow problems. Problems I and II differ only in minor respects from the problem treated in [1]. This enables us to reduce these problems to such a form that the results of interest can be obtained directly in terms of quantities computed in that paper. The ordinary differential equation which χ, the Fourier transform of ψ, [i.e. $\chi(\xi, y) = \int_{-\infty}^{\infty} \psi(x, y) \exp(-i\xi x)\, dx$], must obey is

$$y^{-1}(y\chi_y)_y + (k^2 - \xi^2)\chi = 0. \tag{3.1}$$

The appropriate solution (obeying the radiation condition) is

$$\chi = A(\xi)J_0([k^2 - \xi^2]^{1/2}y), \qquad y < 1$$

and

$$\chi = B(\xi)H_0^{(1)}([k^2 - \xi^2]^{1/2}y), \qquad y > 1. \tag{3.2}$$

The boundary conditions can be written concisely if we define $h(\xi) = (ik - iM\xi)[\chi(\xi, 1-) - \chi(\xi, 1+)]$. The pressure boundary condition then merely implies that $h(\xi)$ is the transform of a function, $H(x)$, which is zero for $x > 0$. We also note that $w(\xi) = \chi_y(\xi, 1)$ is the transform of a function, $W(x)$ which vanishes for $x < 0$. Now

$$h(\xi) = i(k - M\xi)\{A(\xi)J_0([k^2 - \xi^2]^{1/2}) - B(\xi)H_0^{(1)}([k^2 - \xi^2]^{1/2})\}$$

and

$$w(\xi) = -A(k^2 - \xi^2)^{1/2}J_1([k^2 - \xi^2]^{1/2}) = -B(k^2 - \xi^2)^{1/2}H_1^{(1)}([k^2 - \xi^2]^{1/2}).$$

Eliminating A and B, we obtain (using $J_1(\zeta)H_0^{(1)}(\zeta) - J_0(\zeta)H_1^{(1)}(\zeta) = 2i/\pi\zeta$),

$$-\pi i(k^2 - \xi^2)H_1^{(1)}([k^2 - \xi^2]^{1/2})J_1([k^2 - \xi^2]^{1/2})h(\xi) = 2i(k - M\xi)w(\xi). \tag{3.3}$$

Our foregoing remarks imply that $h(\xi)$ is analytic when $\operatorname{Im} \xi > 0$ and that $w(\xi)$ is analytic

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[1] We consider only problems in which the incident wave is a plane wave in the tube normal to its axis and approaching the origin.

when Im $\xi < 0$. The domains of analyticity of these functions can be extended if we associate with k a complex value (Im $k = \epsilon > 0$). When this is done, we can anticipate that h, w, and the coefficient of h in (3.3) will be analytic in the strip $|$ Im ξ $| < \epsilon$. This is demonstrated in detail in [1]. We now write[2]

$$L(\xi) = \pi i H_1^{(1)}([k^2 - \xi^2]^{1/2})J_1([k^2 - \xi^2]^{1/2}) = L_+(\xi)/L_-(\xi),$$

where L_+ and L_- are respectively analytic above $-i\epsilon$ and below $i\epsilon$. One can now apply the conventional arguments of the Wiener-Hopf technique to find $h(\xi)$, and these arguments lead to the conclusion that each side of the equation

$$-(k^2 - \xi^2)L_+(\xi)h(\xi) = 2i(k - M\xi)L_-(\xi)w(\xi) \tag{3.4}$$

is the analytic continuation of the other, and that each represents an entire function of ξ.

When $M > 0$ (the exit problem), it is required that $H(x) \to 0$ as $x \to 0-$ and we can anticipate that $H(x) \sim |x|^{1/2}$ in this neighborhood. It was shown in [1] that $L_+(\xi) \sim \xi^{-1/2}$ for large ξ (with Im $\xi > 0$). It follows that the entire function, $(\xi^2 - k^2)L_+(\xi)h(\xi)$, is a constant (because it is bounded at ∞).

When $M < 0$, the situation is different. In this case, instead of demanding that $h(x) \to 0$ as $x \to 0-$, we require that ψ be continuous on the extension of the tube. This implies that $-ih/(k - M\xi)$ be the transform of a function which behaves like $x^{1/2}$ near $x = 0$ and, in this case, the entire function $(k^2 - \xi^2)L_+(\xi)h(\xi)$, is a constant multiple of $(k - M\xi)$. One also can see this by noting that, in the entrance case, the radial velocity perturbation near the edge must behave like $x^{-1/2}$; then, since $L_-(\xi) \sim \xi^{1/2}$ (see [1]), the foregoing result ensues.

We are primarily interested in the reflection coefficient associated with the tube end. Since the pressure field in the tube for large negative x must be of the form $H(x) \sim \exp(iks) + N \exp(-iks)$, it follows that the reflection coefficient N is the ratio of the residues of $h(\xi)\exp(i\xi s)$ at $-k$ and $+k$, respectively (i.e. $N = -L_+(k)/L_+(-k)$ for the exit problem and $N = -[(1 + M)/(1 - M)]L_+(k)/L_+(-k)$ for the inlet problem, where $M < 0$). For the exit problem, these residues are precisely those computed in [1]. Thus our $N[(\omega R/a)(1 - M)^{1/2}]$ is precisely the R of [1]. Note that this is the reflection coefficient for the pressure field (and, as it happens, for the velocity field, too) but *not* for the potential.

For the inlet flow, on the other hand, the residues of $h(\xi)\exp(i\xi s)$ at $\pm k$ are not the same as those of the exit flow but rather those of this expression multiplied by $i(k - M\xi)$. Thus, for $M < 0$, the reflection coefficient $N'(k)$ is given by $[(1 + M)/(1 - M)]N(k)$.

It is interesting to note that the reflection at the inlet end of the tube is much poorer than that at the exit end. This is most readily rationalized by noting that, at the inlet end, the incident wave "sees" a much smaller wave length to radius ratio. This is not an adequate basis for an estimate, however, because the non-convective theory gives a quadratic dependence on this ratio for small frequency whereas here the dependence is stronger and, in particular, N is not unity for $\omega = 0$.

The major result is to notice that for low frequencies, a moving tube has much greater radiation losses from an inlet end than it does in a stationary situation but that the losses at the exit are only affected when M gets reasonably close to unity.

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[2] The notation is nearly identical with that of [1] in order that the reader may readily translate the thorough discussion of that paper into the straightforward generalization of this section.

Figures (3.1) and (3.2) are graphs of N and l/R where $N = -| N | \exp (2ikl/R)$. These are taken from [1].

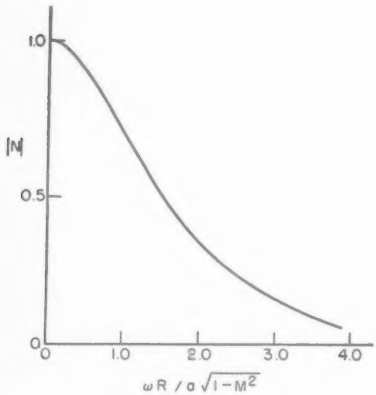

FIG. 3.1.

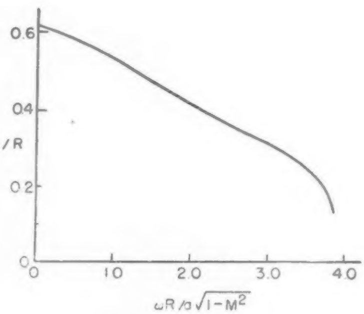

FIG. 3.2.

4. The jet flow. In this section we consider the algebraically more difficult problem where $M_1 \neq M_2$ and $a_1 \neq a_2$. In this problem, it is more convenient to distort the coordinate system uniformly (i.e. we use the same scale for x and r) and we introduce the notation: $s = x/R$, $y = r/R$, $m_i = v_i/a_i$, $k_i = \omega R/a_i$, $\beta_i^2 = 1 - M_i^2$ and $\mathbf{v}_i = a_i$ grad $\phi_i(s, y) \exp(-i\omega t)$. The differential equations governing the ϕ_i become (proceeding from Eqs. (2.1) and (2.2))

$$\Delta\phi_1 - \left(M_1 \frac{\partial}{\partial s} - ik_1\right)^2 \phi_1 = 0, \tag{4.1}$$

$$\Delta\phi_2 - \left(M_2 \frac{\partial}{\partial s} - ik_2\right)^2 \phi_2 = 0. \tag{4.2}$$

If we again introduce the Fourier transforms with regard to x of the ϕ_i, call them χ_i (ξ, y), and integrate the ordinary differential equations which govern the χ_i (ξ, y), we obtain

$$\chi_1(\xi, y) = A(\xi)J_0(z_1 y), \tag{4.3}$$

$$\chi_2(\xi, y) = B(\xi)H_0^{(1)}(z_2 y), \tag{4.4}$$

where $z_i^2 = (k_i - M_i\xi)^2 - \xi^2$. The velocity at $y = 1$, $W(s)$ has the transform

$$w(\xi) = -z_1 A J_1(z_1) = -z_2 B H_1^{(1)}(z_2) \tag{4.5}$$

and the jump[3] in pressure at $y = 1$, $H(s)$, has the transform

$$h(\xi) = (k_1 - M_1\xi)AJ_0(z_1) - (k_2 - M_2\xi)BH_0^{(1)}(z_2). \tag{4.6}$$

Equations (4.5) and (4.6) may be combined to eliminate A and B and to obtain

$$z_1^2 L(\xi)h(\xi) = w(\xi)[\alpha - (M_1 + M_2\beta_1/\beta_2)\xi], \tag{4.7}$$

[3]This is the dimensionless jump $(p_1' - p_2')/p_1$. Note that $p_1 = p_2$ for equilibrium of the steady flow.

where

$$L(\xi) = \frac{[\alpha - (M_1 + M_2\beta_1/\beta_2)\xi](z_2/z_1)J_1(z_1)H_1^{(1)}(z_2)}{(k_2 - M_2\xi)z_1H_0^{(1)}(z_2)J_1(z_1) - (k_1 - M_1\xi)z_2H_1^{(1)}(z_2)J_0(z_1)} \tag{4.8}$$

and where α is so chosen that $L(\xi)$ is finite at that zero of the denominator of Eq. (4.8) which, when states 1 and 2 are identical, is located at $\xi = k_2/M_2$.

It is convenient to factor $L(\xi)$ in the following manner.

$$L_1(\xi) = \pi i J_1(z_1)H_1^{(1)}(z_1), \qquad L^*(\xi) = L/L_1,$$

$$= \frac{-i[\alpha - (M_1 + M_2\beta_1/\beta_2)]z_2H_1^{(1)}(z_2)}{\pi[(k_1 - M_1\xi)z_1J_0(z_1)H_1^{(1)}(z_1)z_2H_1^{(1)}(z_2) - (k_2 - M_2\xi)H_0^{(1)}(z_2)z_1^2H_1^{(1)}(z_1)J_1(z_1)]} \tag{4.9}$$

and the factor $L_1(\xi)$ is precisely the Schwinger-Levine $L(\zeta)$ where $\zeta = \beta_1\xi + k_1$.

The solution of Eq. (4.7) for $h(\xi)$ now requires that we split $L(\xi)$ [and hence both $L_1(\xi)$ and $L^*(\xi)$] into factors $L_+(\xi)$, $L_-(\xi)$ which are analytic above $-i\epsilon_2$ and below $i\epsilon_1$, respectively (here ϵ_i is the imaginary part of the root of $z_i = 0$ when $\text{Im } k_2 = \epsilon$).

The reflection coefficient will be given again by the ratio of the amplitudes of the returning and the outgoing waves in the tube. The inversion integral for the pressure $H(x)$ in the tube (as $x \to -\infty$, the external pressure tends to 0 so the interior pressure becomes the pressure jump $H(x)$) is given by

$$H(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [L_+(\xi)]^{-1} \exp(i\xi x)[z_1(\xi)]^{-2}\, d\xi$$

and the asymptotic result for $-x \gg 1$ is given by the residues at the zeros of z_1^2. That is, the reflection coefficient is again $L_+(-b_2)/L_+(b_1)$ where $-b_2$ and b_1 are the zeros of z_1^2. However, L_+ is composed of two factors, the $L_+(\xi)$ associated with L_1 as used in the foregoing and the factor L_+^*. The Cauchy integral formula gives L_+^* in the form

$$\ln L_+^*(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\xi - t)^{-1} \ln L^*(\xi)\, d\xi \tag{4.10}$$

and, in particular, we define the quantity Q so that

$$\ln Q = \ln(L_+^*(-b_2)/L_+^*(b_1)) = \frac{-k_1}{\pi i\beta_1^2}\int_{-\infty}^{\infty} \frac{\ln L^*(\xi)}{(\xi - b_1)(\xi + b_2)}\, d\xi, \tag{4.11}$$

where the path of integration is indented to pass below $-b_2$ and b_1. When $k_1 = k_2$, $M_1 = M_2$, $L^*(\xi)$ is identically unity and the contribution of (4.11) is $Q = 1$. For other cases, however, $L^*(\xi) \to 1$ as $\xi \to \pm\infty$ and the convergence of the integral is assured. The reflection coefficient in such cases is given by the product

$$Q(k_1, k_2, M_1, M_2)N(k_1/\beta_1).$$

Again one can use the results of [1] to find N but the determination of Q requires the numerical evaluation of the integral of (4.11) for those values of the four independent parameters a_1, a_2, M_1, M_2 which are of interest.

Bibliography

1. H. Levine and J. Schwinger, *On the radiation of sound from an unflanged circular pipe*, Phys. Rev. **73** (1948)

SOLUTIONS OF THE HYPER-BESSEL EQUATION*

By CHIA-SHUN YIH, (*State University of Iowa*)

In problems of hydrodynamic stability involving axial symmetry, it is sometimes necessary to find the solutions of a differential equation of the type

$$L_1^n f = 0$$

in which n is a positive integer, and (with $D \equiv d/dr$)

$$L_1 \equiv D^2 + r^{-1}D - r^{-2} - \lambda^2$$

is the Bessel operator of the first order. In this note, solutions of the equation

$$L_p^n f = 0 \tag{1}$$

in which

$$L_p \equiv D^2 + r^{-1}D - p^2r^{-2} + k^2$$

will be given explicitly. The theorem one seeks to establish is the following: If p (taken to be positive for convenience) is not an integer, the solutions of Eq. (1) are $r^m J_{\pm(p+m)}(kr)$ in which $m = 0, 1, 2, \cdots, n-1$; otherwise they are $r^m J_{p+m}(kr)$ and $r^m N_{p+m}(kr)$, with m ranging over the same integers. The symbols J and N stand for the Bessel function and the Neumann function, respectively.

Proof: It is known that the solutions of $L_p f = 0$ are $J_{\pm p}(kr)$ for p not equal to an integer and $J_p(kr)$ and $N_p(kr)$ for p equal to an integer. Thus it suffices to show that if $r^s Z_{p+s}(kr)$ (in which Z stands for either J or N) satisfies $L_p^{s+1} f = 0$, then $r^{s+1}Z_{p+s+1}(kr)$ satisfies $L_p^{s+2} f = 0$, since the proof for $r^m J_{-(p+m)}(kr)$ is identical with that for $r^m Z_{p+m}(kr)$. This will be accomplished if one can show that $L_p r^{s+1} Z_{p+s+1}(kr)$ is equal to a constant times $r^s Z_{p+s}(kr)$. By straightforward differentiation one has

$$\begin{aligned} L_p r^{s+1} Z_{p+s+1}(kr) &= r^{s+1} L_p Z_{p+s+1}(kr) + s(s+1)r^{s-1} Z_{p+s+1}(kr) \\ &\quad + 2(s+1)r^s DZ_{p+s+1}(kr) + (s+1)r^{s-1}Z_{p+s+1}(kr) \\ &= r^{s+1} L_p Z_{p+s+1}(kr) + (s+1)^2 r^{s-1} Z_{p+s+1}(kr) \\ &\quad + 2(s+1)r^s DZ_{p+s+1}(kr). \end{aligned}$$

But

$$L_p = L_{p+s+1} + \frac{2p(s+1) + (s+1)^2}{r^2}$$

and [1]

$$DZ_{p+s+1}(kr) = k\left[-\frac{p+s+1}{kr} Z_{p+s+1}(kr) + Z_{p+s}(kr) \right].$$

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*Received May 14, 1955.

So

$$L_p r^{s+1} Z_{p+s+1}(kr) = r^{s+1} L_{p+s+1} Z_{p+s+1}(kr) + [2p(s+1) + (s+1)^2] r^{s-1} Z_{p+s+1}(kr)$$

$$+ (s+1)^2 r^{s-1} Z_{p+s+1}(kr) + 2(s+1) r^s k \left[-\frac{p+s+1}{kr} Z_{p+s+1}(kr) + Z_{p+s}(kr) \right]$$

$$= 2(s+1) k r^s Z_{p+s}(kr)$$

since

$$L_{p+s+1} Z_{p+s+1}(kr) = 0$$

by definition of Z.

Dr. Y. C. Fung of the California Institute of Technology communicated to the writer a different proof of the present result by means of Almansi's theorem [2] on hyper-harmonic functions. His proof will not be presented here.

It may be noted that since [1]

$$Z_{p-1}(kr) + Z_{p+1}(kr) = \frac{2p}{kr} Z_p(kr) \tag{2}$$

and since by the theorem just proved $rZ_{p+1}(kr)$ and $Z_p(kr)$ are solutions of

$$L_p^2 f = 0, \tag{3}$$

it follows from Eq. (2) that $rZ_{p-1}(kr)$ is also a solution of Eq. (3). In fact, by repeated use of Eq. (2) and a similar one obtained by changing p to $-p$ in Eq. (2), it can be proved that if the m in the subscripts of the solutions given in the theorem is changed to $-m$, the results will still be solutions of Eq. (1). These solutions are of course not independent of the ones given in the statement of the theorem.

References

[1] E. Jahnke and F. Emde, *Table of functions*, Dover Publications, New York, 1945, pp. 144-145

[2] E. Almansi, *Sull' integrazione dell' equazione differenziale* $\Delta^{2n}u = 0$, Annali di Matimatica, (III) 2 (1899)

BOOK REVIEWS

(*Continued from p. 392*)

Mathematics of engineering systems. By Derek F. Lawden. Methuen & Co., Ltd., London, and John Wiley & Sons, Inc., New York, 1954. viii + 380 pp. $5.75.

This book provides a course in applied mathematics in which a selection of topics of interest in modern engineering and applied science are treated on an intermediate level. After an introductory chapter which includes complex numbers there are two chapters on linear ordinary differential equations with constant coefficients. The classical methods are treated in the first of these, with applications to problems of servomechanisms and stability; modern methods are taken up in the second, involving the use of standard inputs (unit steps, unit impulse, and sinusoidal functions) and the Laplace transform. Chapter 4 is concerned with Fourier series and integrals and applications of Fourier transforms. The final chapter provides a brief introduction to non-linear differential equations, in which main attention is paid to Van der Pol's and to Duffing's equation. This is a book of mathematics, not of engineering; but the large number of examples refer to practical engineering problems. They are drawn mostly from electrical engineering (electronic amplifiers and oscillators, electric circuits, and servomechanisms), but the book should be of interest to engineers and scientists in many fields.

P. S. Symonds

La Théorie des Fonctions de Bessel. By Gerard Pétiau. Renseignements et Vente au Service des Publications du CNRS, Paris, 1955. 477 pp. $7.20.

This book would be a very valuable addition to the literature on Bessel functions if it were not lacking some rather vital features. The most serious omission is an alphabetical index. This is seldom of much value in trying to find an equation but it does serve an important function in finding some things and the lack of one seriously reduces the usefulness of the book. The book contains references for most topics of recent development and for certain topics of related interest to the main topics but there is no general list of references and by and large the system of references is rather weak. The book is also lacking a preface; thus there is nowhere in the book any discussion of the scope or extent to which this book supersedes previous works on Bessel functions or even for what purpose the book was intended.

On the more favorable side, the book does contain a tremendous collection of formulas. Most of the derivations are short and the book seems to be arranged with this particularly in mind. Many formulas and even some interesting topics are not to be found in some of the standard references (for example Watson's "The Theory of Bessel Functions" or the recent Bateman Manuscript Project "Higher Transcendental Functions") There is a short chapter on Bessel integral functions

$$Ji_n(x) = -\int_x^\infty u^{-1} J_n(u)\, du$$

and related integrals. There are also some definite integrals that are not easily found elsewhere.

The scope of the book is extensive. The first 328 pages are devoted to the properties of Bessel functions and related functions. It includes much of the material in Watson's treatise plus some more recent topics. The next 107 pages describes some of the principal applications particularly in relation to the solution of partial differential equations. By comparison this part is much more illustrative than exhaustive. In addition there are 14 pages of short tables and 11 pages of graphs.

The derivations of some of the formulas employ standard mathematical techniques which are of a somewhat advanced nature. This, coupled with the scarcity of references, will make the book difficult to follow for a person not well trained in mathematics.

G. F. Newell

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